THE
ASTROPHYSICAL JOURNAL
AN INTERNATIONAL REVIEW OF SPECTROSCOPY
AND ASTRONOMICAL PHYSICS

VOLUME XXXVI DECEMBER 1912 NUMBER 5

ON THE DEVIATION FROM LAMBERT'S COSINE LAW OF THE EMISSION FROM TUNGSTEN AND CARBON AT GLOWING TEMPERATURES

By A. G. WORTHING

INTRODUCTION

Lambert's cosine law as applied to the radiation from a self-luminous body is expressed by the following equation

$$b \equiv \frac{dI}{dS \cos\theta} = b_0$$

where b, dI, and dS refer respectively to the brightness, the luminous intensity, and the area of an element of the body when viewed by light emitted at an angle θ, and where b_0 is the brightness viewed normally. Where the cosine law does strictly apply, the brightness does not vary with the angle of emission.

For a study of the theoretical foundations and a survey of the literature relating to Lambert's law, reference is made to Uljanin,[1] Helmholtz,[2] Liebenthal,[3] and Gross.[4] It is sufficient to say here that theoretical considerations seem to show that Lambert's cosine law is rigorously applicable only to a black body.

[1] *Wied. Ann.*, N.F., **62**, 528, 1897. [2] *Vorlesungen*, **6**, 150.

[3] *Praktische Photometrie* (Vieweg & Sohn), pp. 80–88.

[4] *Jahresbericht der Schlesischen Gesellschaft für vaterländische Kultur* (Naturw. Sektion), March 6, 1911.

Relative to the application of the cosine law to the emission of light by a self-luminous body, the earliest data seem to be those by Möller.[1] He investigated the light emitted by a glowing strip of platinum at various angles, making use of a polarization photometer. He concluded from his work that the cosine law of emission was fulfilled. It is to be noted, however, that his actual results point to a consistent and slight deviation therefrom, the deviation being in the direction to make the brightness of the strip appear greater at large angles of emission than at small angles. This variation was ascribed to experimental errors.

Later Uljanin,[2] making use of the optical constants obtained by Drude, computed, for various angles of emission for platinum, silver, and copper, the intensities of the beams of light polarized respectively in the plane and perpendicular to the plane of emission. From these he computed further the polarization of the emitted light and the brightness of the glowing surface as functions of the angle of emission. His computations as to the polarization of the light from platinum are closely verified by his own direct observations, in which, however, he made use of wave-lengths in the neighborhood of $3\,\mu$, whereas the Drude constants were obtained for light in the visible spectrum. Polarization measurements made by Provostaye and Desains[3] and by Millikan[4] agree closely with his results. Observations by Violle[5] and by Millikan[6] verify the polarization computations on silver. These verifications of the polarization computations lend credence to the computed brightness variations. Uljanin computed that the brightness of glowing platinum should increase from the normal value for zero angle of emergence to 1.17 times the normal brightness for an angle of 70°, decreasing at somewhat larger angles. It was suggested that Möller's observations were undoubtedly in error due to some dust which may have collected on the surface or due to the roughening of the surface of the platinum consequent on heating it up to glowing temperatures. This latter suggestion was rendered plausible by further experi-

[1] *Ann. d. phys.*, N.F., **24**, 266, 1885.
[2] *Loc. cit.*
[3] *Ann. de chim. et de phys.* (3), **32**, 112, 1851.
[4] *Physical Review*, **3**, 81, 177, 1895.
[5] *Comptes Rendus*, **105**, 111, 1886.
[6] *Loc. cit.*

mentation in which the strip originally used by Uljanin to verify his polarization computations was heated to incandescence and then cooled somewhat. The further polarization measurements showed a considerable decrease in the polarization for any given angle of emission.

Recently in a study of the energy losses in electric incandescent lamps, in collaboration with Messrs. Hyde and Cady,[1] the writer noted certain variations which have led to the investigation described in the present paper.

APPARATUS AND METHOD

The arrangement of the apparatus as shown in Fig. 1 is the same as that described in the energy-loss paper just referred to. The materials used in this investigation were large lamp filaments

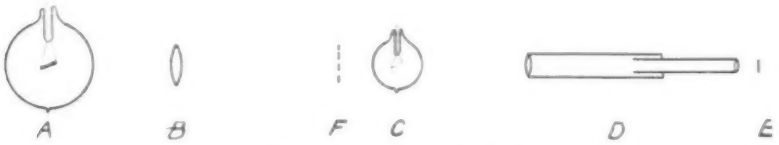


FIG. 1.—Diagrammatic sketch of apparatus

mounted in spherical bulbs, the portions used being those near the centers of the bulbs. *A* represents such a lamp; *B*, a lens so placed as to produce an enlarged image of the filament of *A* in the plane of a thin filament mounted in another bulb at *C*. *D* represents a telescope; *E*, a piece of red or blue glass. The transmissions of the glasses used are such that the light transmitted may for computational purposes be considered monochromatic, the wavelengths being respectively 0.63 μ and 0.46 μ. *F* represents an absorbing glass screen which, though not regularly used, was used in the special case when the effect of changes in temperature was investigated.

There are two ways of manipulating lamp *A* in order that the variations of brightness with the angle of emission may be investigated. The one used for the most part is that of rotating lamp *A* about a vertical axis passing through the point investigated. The

[1] *Trans. Ill. Eng. Soc.*, **6**, 238, 1911; *Ill. Eng.* (Lond.), **4**, 389, 1911.

other way consists in raising or lowering lamp A so that different portions of a cross-section of A are seen in the line with the filament of lamp C. The method is purely an optical pyrometric one. It consists in finding for various angles of emission for the light from lamp A the various voltages of lamp C at which it matches in intensity the portions of the image of lamp A which are in close juxtaposition. From such data one may obtain the variations in brightness of lamp A with angle of emission by determining the relative intensities of lamp C for the red or blue light of the desired wave-lengths for the different voltages obtained.

This has been accomplished by means of ordinary photometric measurements in which first the red glass and later the blue glass used at E (Fig. 1) were placed at the ocular opening of the Lummer-Brodhun contrast photometer-head which was used. No appreciable color differences were noticed. From a plat of the logarithm of the voltage as a function of the logarithm of the candle-power of lamp C, there was obtained, by means of a straight-edge, a quantity k_λ, the limiting value of the ratio of a small relative change in luminous intensity to the corresponding relative change in voltage, as a function of the lamp voltage. The results of such measurements have been plotted in Fig. 2. It is to be noticed that for any given voltage the ratio of the k_λ for the red light to the k_λ for the blue light is very closely $\frac{4}{6}\frac{6}{3}$, the inverse of the ratio of the effective wave-lengths of transmission. This condition is rigorously fulfilled for black-body radiation when the two colored lights are each monochromatic. Since at any one temperature, as shown by Hyde,[1] the light from a tungsten lamp very closely matches spectrophotometrically that from a black body at some certain temperature, it follows that this condition should very closely apply to a tungsten lamp, as it does. The application of the curves of Fig. 2 to determine the brightness of the filament of lamp A for various angles of emission is evident.

The question of a change in the variation from the cosine law with temperature was especially suggested by the paper already referred to by Dr. Hyde. In order to make more certain whether such a change existed or not, a slight modification of the above-

[1] *Astrophysical Journal*, **36**, 89, 1912.

described method was undertaken. It consisted in taking voltage readings of lamp *C* in the ordinary way for angles of emission from lamp *A* of 0° and of 75° for a low operating temperature and then with an absorbing glass screen at *F* for a high operating temperature such that in the second instance the voltages for lamp *C* were approximately the same as those for the first instance. The two readings at 75° were made successively without any other change than the introduction or the removal of glass *F*.

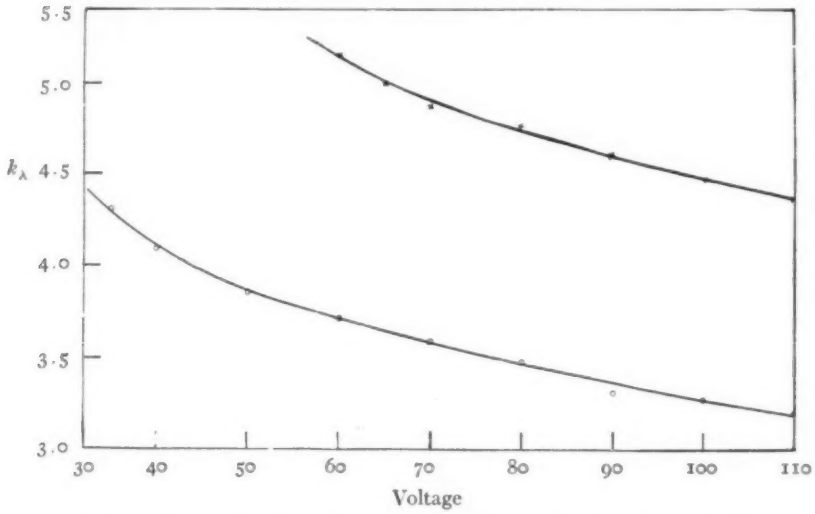


FIG. 2.—k_λ as a function of the voltage for lamp *C*, where k_λ is defined as the limiting value of the ratio of a small relative change in luminous intensity for the wave-length λ to the corresponding relative change in voltage.

× λ=0.46 μ o λ=0.63 μ

RESULTS ON BRIGHTNESS MEASUREMENTS

These have been determined, using light of wave-length 0.63 μ, the brightness as a function of the angle of emission for tungsten at two different temperatures, one being that corresponding to the normal operating temperature with an efficiency of 1.25 watts per candle-power and the other to a temperature at which the wattage of the lamp is about 18 per cent of the normal, and for carbon at one temperature. The individual determinations together with the smoothed curves are shown in Figs. 3, 4, and 5.

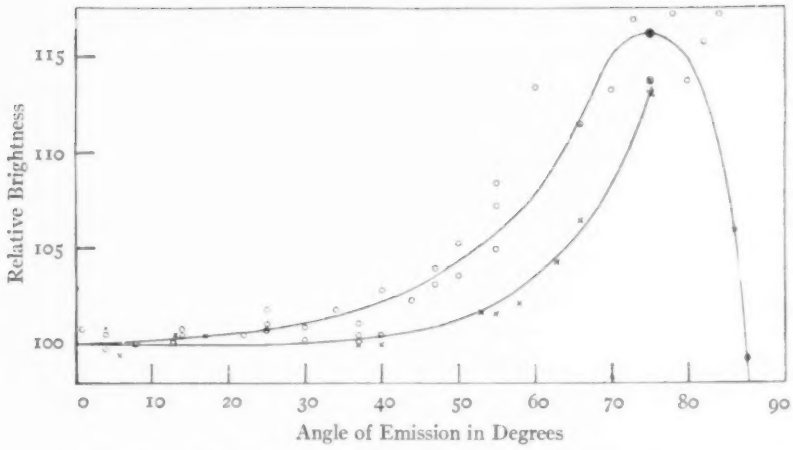


FIG. 3.—The deviation from Lambert's law of the emission from tungsten at normal operating temperature.

o $\lambda = 0.63\ \mu$ × $\lambda = 0.46\ \mu$

• $\lambda = 0.63\ \mu$ average of several values

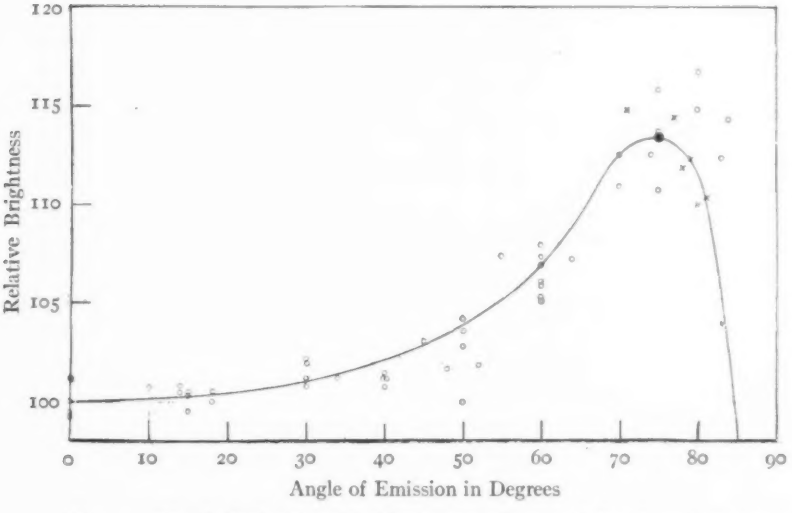


FIG. 4.—The deviation from Lambert's law of the emission from tungsten at a low operating temperature for light of wave-length $0.63\ \mu$.

• Average of several values

× Values obtained when special care was taken to eliminate errors creeping in at large angles.

A few values for the variation of brightness with angle of emission have been determined for tungsten at the higher temperature, using light of wave-length 0.46 μ, and for carbon at one temperature. In Fig. 5 there are plotted data obtained on two different lamps, both untreated carbon. The data for the red light, indicated by crosses (×), were obtained on the same lamp filament as were the data for the blue light. Because of a desire to compare

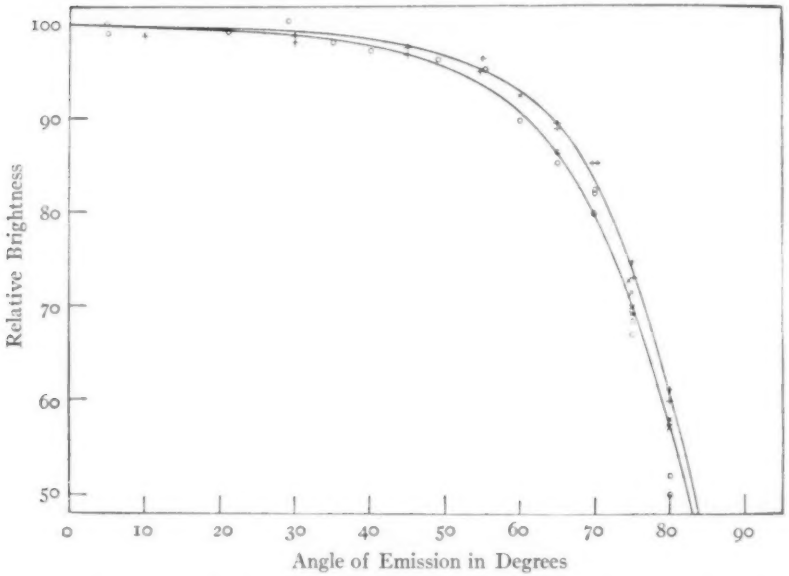


FIG. 5.—The deviation from Lambert's law of the emission from incandescent carbon.

o λ=0.63 μ first lamp studied + λ=0.46 μ second lamp studied
× λ=0.63 μ second lamp studied
Many individual values in the neighborhood of 0° have been omitted from the plat

the blue- and the red-light variations, the data plotted as crosses have been given preference to those plotted as circles. The large variation in the individual results in connection with tungsten is due largely to three facts: (1) The axis of rotation of lamp *A* (Fig. 1) was not tangent to the filament. With each new angle of emission there was a consequent slight apparatus adjustment. (2) The glass of the bulb was not uniform. (3) The laboratory work was

rendered difficult, due to the close location to the city streets. The difference, however, between the variations for the blue light and the red light is real for both tungsten and carbon.

In the investigation of the effect of temperature on the variations, seven sets of determinations showed that for red light tungsten does change with temperature. The average of these gave for 75° an increase in the deviation with the increased temperature amounting to 22 per cent, with an average variation from it of 5 per cent. The two operating temperatures were approximately those which were used in the regular determination of the variation of brightness with angle of emission. The results have been plotted on the curves given as filled circles at 75°. In finding the variation with temperature for blue light the various brightnesses for the blue light were directly compared with the corresponding brightnesses for the red light. Some of the experimental uncertainties which have been stated elsewhere were thus eliminated. The change with temperature for the blue light was found to be very closely the same as for the red light, the exactness of agreement falling well within the range of experimental error. The single value plotted in Fig. 3 for the blue light at 75° was obtained from these measurements, and is dependent on the correctness of the value plotted for the red light. No such changes with temperature were found for carbon.

The making of brightness determinations is difficult for large angles of emission. In Fig. 3 there are plotted as crosses values obtained for large angles of emission for which special care was taken to eliminate as many errors as possible. Two values for the relative brightness 61 and 78 for angles of emission greater than 86° were obtained but have not been included in the plat. By making use of a U-shaped filament, this decrease in brightness for large angles may be readily observed if the filament is so rotated about a central vertical axis that the two limbs are nearly in the line of sight. A fairly large portion of the lower curved portion emitting light at a large angle will then be found in close juxtaposition to fairly large areas emitting light at smaller angles of emission. In Fig. 6 there has been plotted the brightness of cylindrical glowing filaments of tungsten and carbon at normal operating

temperatures as functions of the apparent distances from the central axes when viewed at a distance.

The following conclusions may be drawn from these plats:

1. The brightness of tungsten beginning with normal emission increases with the angle of emission, reaches a maximum at about 75°, and for somewhat larger angles decreases rapidly. The brightness of carbon, beginning with normal emission, decreases with

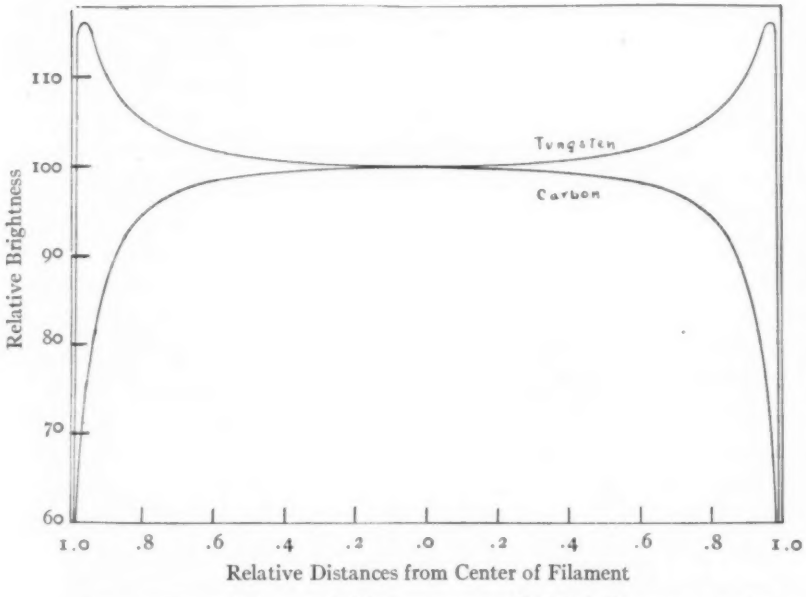


FIG. 6.—The variations in brightness across cylindrical filaments of carbon and tungsten as viewed from a distance.

increasing angle of emission, the rate of decrease increasing with the angle.

2. The relative brightness variations for light of short wave-length 0.46 μ for tungsten are noticeably less than those for the light of the longer wave-length 0.63 μ. The same effect is true for carbon, though the difference is less noticeable.

3. The relative brightness variations for tungsten at the higher of the two temperatures chosen is about 20 per cent or 25 per cent greater than the corresponding variations for the

lower temperature. No definite change with temperature was found for carbon.

4. The average brightness of a tungsten cylindrical filament, viewed normally to its axis from a distance, is found to be about 3 per cent greater than that of the central part, that for a carbon filament about 5 per cent less.

It is interesting to compare the experimental results with what may be obtained by computation, making use of the optical constants obtained by H. v. Wartenberg.[1] The formulae used in these computations are the approximation formulae given by Drude.[2] A few test applications of the more rigorous formulae[3] made no very apparent changes in these results. The writer is indebted to Dr. I. Langmuir of the Research Laboratory of the General Electric company for such computations relative to the emission from tungsten. Column 1, Table I, gives the angle of emission, columns 2, 3, and 4 the reflectivities for light polarized in the plane of emission, for light polarized normal to the plane of emission, and for natural light. Column 5 contains the relative brightness expressed in terms of the normal brightness, obtained by making use of Kirchhoff's law. Column 6 contains the experimental results obtained by the writer for tungsten at the higher of the two operating temperatures. It is to be noticed that the variations shown in the computed values, in both cases, though generally in the same direction as the observed values, are of a different order of magnitude, excepting perhaps for the large angles of emission. It is not to be forgotten in this connection that the Drude constants were obtained with light of wave-length 0.579 μ, while the results used by the writer in the comparisons were obtained with light of wave-length 0.63 μ.

POLARIZATION TESTS

The arrangement of apparatus for these tests is quite simple. By means of a lens an enlarged image of the filament investigated was formed on a piece of ground glass just in front of a Marten's

[1] *Verh. Deutsch. phys. Ges.*, **12**, 105, 1910.

[2] Winkelmann, **6**, 1300.

[3] Chwolson (German ed.), **2**, 880. (In equation [60] the first sign in the numerator should be interchanged with the first in the denominator.)

polarization photometer.[1] Sometimes a piece of colored glass, either red or blue, was placed just back of the ocular opening. The photometer as used had the disk with a single opening at the end next to the ground glass.

TABLE I

COMPUTED AND OBSERVED BRIGHTNESSES AND POLARIZATIONS OF TUNGSTEN ($n=2.76$, $nx=2.71$) AND OF CARBON ($n=2.98$, $nx=1.74$) AS FUNCTIONS OF THE ANGLES OF EMISSION

ANGLE	REFLECTIVITY			RELATIVE BRIGHTNESS		POLARIZATION	
	R_p	R_s	R	Comp.	Obs.	Comp.	Obs.
			TUNGSTEN				
0°	0.486	0.486	0.486	1.000	1.000	0.000	.....
15	.497	.474	.486	1.000	1.003	.022	.....
30	.533	.437	.485	1.002	1.011	.093	.....
45	.597	.370	.484	1.004	1.03	.220	0.156
60	.693	.264	.478	1.015	1.08	.411	.285
75	.826	.167	.497	.978	1.16	.655	.458
80	.880	.205	.542	.891	1.15	.737	.520
85	.938	.391	.664	.654	1.08	.813	.585
90	1.000	1.000	1.000	.000	.....	.875	.....
			CARBON				
0°	0.367	0.367	0.367	1.000	1.000	0.000	.....
15	.379	.354	.366	1.001	.996	.020	.....
30	.418	.314	.366	1.002	.989	.082	0.022
45	.492	.245	.368	.998	.968	.196	.054
60	.605	.142	.374	.989	.906	.370	.096
70	.702	.079	.390	.964	.80	.511	.127
75	.772	.078	.425	.908	.70	.603	.144
80	.840	.133	.486	.812	.58	.689	.160
85	.917	.354	.636	.522	.....	.774	.....

The computed values are for $\lambda=0.579\ \mu$, the observed for $\lambda=0.63\ \mu$.

The first thing to be tested was the depolarizing action of the ground glass. This was found to be very small, of the order of 3 per cent for white light and for red light, and a trifle larger perhaps for blue light. There has been investigated for carbon and for tungsten the polarization of the emitted light as a function of the angle of emission. These results are included in Fig. 7. The change in polarization with change in temperature has also been

[1] *Phys. Zeit.*, **1**, 299, 1900.

investigated. Without any other changes the temperature of the filament of lamp *A* was varied over a very large range and polarization measurements were taken for some large angle of emission. No change was found either for carbon or for tungsten. The value plotted in Fig. 7 for tungsten at 76°.5 as a filled circle is the average of 13 measurements made in this connection. Similar tests using

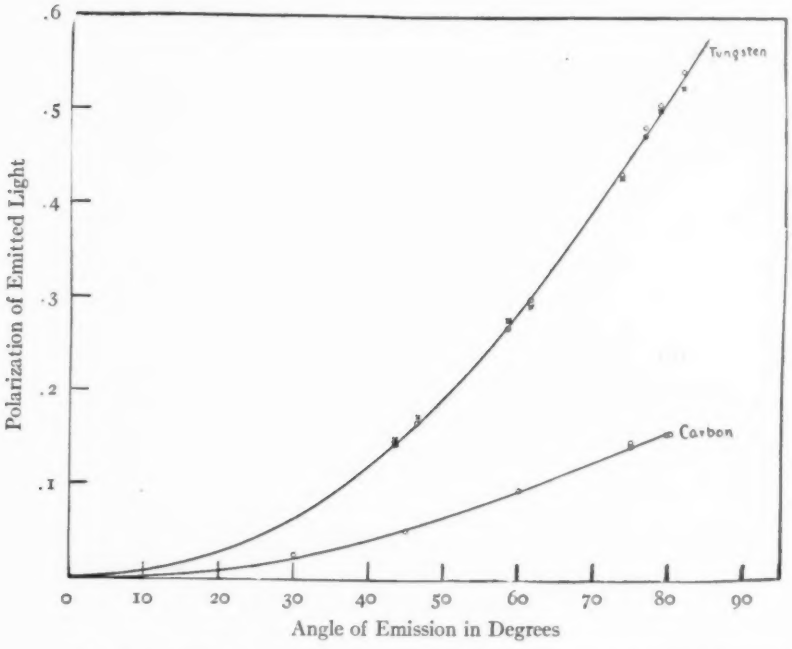


FIG. 7.—The polarization of the light from tungsten and carbon as a function of the angle of emission.

× Natural light o Light of wave-length 0.63 μ

light of different wave-lengths revealed no certain change. Because of this fact and because of the greater ease with which measurements could be made using the natural light from lamp *A*, such measurements have been included in Fig. 7.

It is interesting to compare these results for tungsten and carbon with those which may be computed from the optical constants. In column 7, Table I, will be found such computed values; in column 8 the observed values. It will be noticed for tungsten

that throughout the observed values are very closely 0.7 of the computed values. For carbon the differences between the observed and the computed values are very great.

DISCUSSION OF RESULTS

As sources for the observed discrepancies between the computed and the observed results, there appear at least five facts to be considered.

1. The computed results are based on determinations made with light of wave-length 0.579 μ, while the observed data are for light of wave-lengths approximately 0.63 μ and 0.46 μ.

2. The results of Wartenburg were obtained for materials at ordinary temperatures, while the writer's were obtained with the materials at incandescent temperatures.

3. The existence of a temperature gradient in the incandescent filament.

4. The roughness of the filament surface.

5. The existence of an anisotropic surface layer.[1]

The reflectivity data on both carbon and tungsten by Coblentz[2] would indicate that, whatever variations in the optical properties there might occur with a change in wave-length in passing from the red to the blue end of the spectrum, the optical properties for an intermediate wave-length would possess corresponding intermediate values. If such is the case the discrepancies cannot be ascribed to the first of the above-named sources, for the computed deviations are not intermediate between the red and the blue observed deviations. It should be stated here that the reflectivity to be computed from Wartenberg's data[3] on graphite, viz., 36.7 per cent, differs

[1] Perhaps a sixth source of discrepancy, brought to the author's attention by Dr. C. E. Mendenhall, should be added. Irregularities in the glass of the bulb of ordinary lamps were found by him to be the source of great uncertainties, and in the present work, might be thought to produce by chance some of the variations noted. However, investigations on different filaments and at different points on the same filament by the two different methods always gave consistent results. Plate-glass windows would undoubtedly have been a great help in eliminating some uncertainties, but would certainly not have changed the order of the observed effects.

[2] *Bulletin of the Bureau of Standards*, **7**, 197.

[3] *Loc. cit.*

from that obtained by Coblentz, 22.3 per cent, and from the values obtained by Aschkinass[1] for gas coke and anthracite, 3.7 per cent and 4.4 per cent.

In view of the fact that a change in the optical properties of tungsten with temperature has been found, it would seem possible that considerable of the variation between the computed and the observed results on this substance might be due to the second cause stated above. As to carbon, the experimental data do not seem to support any such explanation.

The existence of a temperature gradient in the incandescent filament needs to be considered, since the average distance of the source of radiation from the surface of the radiating body changes with the angle of emission. Such a source of error would naturally make the observed values for the cosine law variation too small for the larger angles of emission. It is interesting to see what changes might be expected due to this temperature gradient. An expression given by Angell[2] serves for computation purposes. Some work by the writer not yet published gives for the thermal conductivity of incandescent carbon a quantity of the order of 0.1 $\frac{\text{watt}}{\text{deg.} \times \text{cm}}$. This combined with an input of 7.2 $\frac{\text{watts}}{\text{cm of length of filament}}$ and a filament radius of 0.013 cm actual data for one of the carbon lamps investigated, gives as the temperature difference between the axis of the filament and the surface 0°.003. This would certainly not account for any of the discrepancies. The formula given by Angell, however, assumes that the radiation originates at the surface of the filament. But if, as is undoubtedly the case, the radiation originates in finite depths, there might still exist near the surface a considerable temperature gradient, due to the fact that the elements of volume near the surface receive less radiation from the side toward the surface than do those elements which are farther from the surface. Such an effect, while it might account for the discrepancy as to the cosine law variation for carbon, would act in the wrong direction in the case of tungsten. Since the

[1] *Ann. d. phys.* (4), **18**, 380, 1905.

[2] *Physical Review*, **32**, 421, 1911.

polarization effects are but little if at all dependent on the temperature, such measurements would not be noticeably affected by this supposed cause.

The fact that the filaments were roughened, due either to being heated up to high temperatures or to other causes, does not explain these differences so far as the deviation from Lambert's law is concerned. Any such roughening would tend to make the glowing filament in question behave more like a black body, for which Lambert's law should hold rigorously. It therefore might be expected that the true deviations, if a polished surface could be used, would be greater than those found. That the filaments, especially the carbon filaments, were roughened, was readily visible when viewed by light emitted at large angles. In the making of the optical pyrometric measurements it was fortunately possible to select places which were apparently smooth. Because of this the writer is of the opinion that the observed variations from Lambert's law are of the right order of magnitude. The effect of this roughening might well explain, however, some of the differences between the observed and the computed polarizations. Here the observed values are noticeably less than the computed values. This is to be expected if the roughening tends to make the glowing filaments behave more like a black body. How much may be accounted for in this way is very difficult to say. If it might be assumed that the portions of the filament which appeared through the telescope to be smooth were in fact polished surfaces, the writer believes that the differences between the observed and the computed polarizations could certainly not be accounted for on the supposition that the roughening tends to make the body black.

The existence of an anisotropic surface layer is to be expected as a consequence of surface tension forces. Uljanin[1] has mentioned this in his paper and speaks as though the anisotropism were an established experimental fact. How much this might affect the observed results is a matter of conjecture without further data.

It is of course possible that the various sources of discrepancy additatively combine to explain the difference between the observed and the computed results.

[1] *Loc. cit.*

The present work seems to have a direct bearing on certain conclusions which have been reached by others. Hyde[1] has found that a tungsten-carbon combination does not satisfy what he has for convenience called "Criterion I," and that therefore, taking one as a standard, the other changes in its optical properties on being heated. Though good reasons were advanced for thinking that the change occurred in the tungsten, they were not conclusive. The present work indicates definitely a change with temperature in the optical properties of tungsten. No change has been found for carbon. The greater deviation from Lambert's law for the light of the longer wave-length for tungsten is in perfect accord with the known pronounced emission selectivity for the shorter wave-lengths and the measured reflectivities of Coblentz.[2] The greater emissivity for the blue light would suggest that for the shorter wave-length the approach to black-body radiation is more nearly complete, that consequently there would be a closer approach to Lambert's law, as is the case. Starting with the close agreement of the blue and the red deviations of carbon from Lambert's law, one is led to the conclusion that it behaves like a gray or a black body. The small variation which has been found agrees in direction with the small variation which Coblentz found in the reflectivities. That it approximates closely a gray body has been well known. It is, perhaps, surprising that so great a deviation from the cosine law should be found for carbon, but this does not, so far as the writer can see, in any way theoretically conflict with the supposed grayness. The variations found for tungsten are exactly similar to those computed by Uljanin for *Pt.*, *Ag.*, and *Cu*.

SUMMARY

1. There has been measured the brightness variation as a function of the angle of emission for carbon at one temperature and for tungsten at two temperatures, making use of light of wave-length 0.63 μ; and for tungsten and carbon at one temperature each, making use of light of wave-length 0.46 μ. The conclusions drawn from these are summarized near the end of the subdivision giving the results on brightness measurements.

[1] *Loc. cit.* [2] *Loc. cit.*

2. Measurements of the polarization of the light emitted at various angles of emission have been made for both carbon and tungsten. No certain changes have been found in the polarization due to changes in temperature or changes in the wave-length of the light used.

3. The comparison of the observed with the computed values for both tungsten and carbon show the observed deviation from Lambert's law to be greater than and of a different order of magnitude from the deviations computed; while for the polarization of the emitted light, it has been found that the observed values are considerably less than the computed values.

4. In certain respects the present results have been found to agree qualitatively with the experimental results of Hyde and of Coblentz.

In conclusion the writer desires to express his indebtedness to Mr. F. E. Cady and to Mr. G. Cadisch for aid given in the experimental work.

PHYSICAL LABORATORY
NATIONAL ELECTRIC LAMP ASSOCIATION
CLEVELAND, OHIO
July 1912

THE INTEGRATED SPECTRUM OF THE MILKY WAY[1]

By E. A. FATH

In *Harvard Annals*, **56**, No. 1, is given a study of the distribution of the spectra of over 32,000 stars. Among other results are found: (1) That over 52 per cent of the stars investigated have A-type spectra; (2) The ratio of A-type stars to all other types increases as the brightness decreases; (3) In the Milky Way two-thirds of the stars investigated are of the A-type.

In view of these observations we might seem justified in drawing the following conclusions: (1) The integrated spectrum of our stellar system would be approximately of the A-type; (2) The integrated spectrum of the Milky Way as seen from our point of view would be of the A-type.

For some years the writer had planned to make a study of the integrated spectrum of some of the brighter regions of the Milky Way in order to test the validity of the second conclusion. An opportunity presented itself from March to June of last year, when Director Campbell of the Lick Observatory very kindly loaned the spectrograph which the writer had used for work on the Zodiacal Light. This instrument is described in *Lick Observatory Bulletin*, No. 165.

The region selected for the first trial was the very bright portion of the Milky Way partially bounded by the stars γ, δ, and λ *Sagittarii*, which is the brightest region of the galaxy seen at this latitude.

The method of observation was the simplest possible. The spectrograph alone was used, as the angular diameter of the surface investigated was greater than the angular aperture of the collimator lens. The instrument was mounted in such a way as to permit motion in azimuth and altitude. It was then pointed at the region in question and moved every 10 or 15 minutes to follow the diurnal motion.

[1] *Contributions from the Mount Wilson Solar Observatory*, No. 63.

PLATE XVIII

INTEGRATED SPECTRUM OF THE MILKY WAY

a Spectrum of the Milky Way in *Sagittarius*. May and June 1911. Total exposure $65^h\ 13^m$. Slit, 0·8 mm.

b Spectrum of sky. Slit, 0·1 mm.

The first exposure was begun on the night of March 30 and continued, whenever possible, until May 4. The total exposure on this first plate was 30^h20^m. As the spectrum obtained was not quite as strong as desired a second plate was taken in the interval from May 27 to June 29. The second exposure totaled 65^h13^m. It was of satisfactory intensity.

The general appearance of this spectrum is like that of the sun in that it shows the F, G, H, and K lines and three broad absorption bands to the violet of K. On closer examination, however, certain differences can be noted. There appears to be a bright line at 416 $\mu\mu$ and faint additional absorption lines at 411, 421, and 448 $\mu\mu$. The bright line is just halfway between the first two absorption lines. This raises the question as to whether it may not be merely a portion of the general continuous spectrum which appears especially strong by contrast. The line appears on both plates, while the three faint absorption lines appear only on the stronger of the two. The origin of these additional lines is uncertain. There is neither a bright auroral nor a bright nebular line in the neighborhood of the bright line observed. The absorption line at 411 $\mu\mu$ may possibly be $H\delta$; those at 421 and 448 $\mu\mu$ cannot be even approximately identified. These wave-lengths are probably uncertain by 1 $\mu\mu$ although for two entirely distinct sets of measures made over a week apart the greatest difference is 0.5 $\mu\mu$. This uncertainty is due in a large measure to the very small scale of the spectrum, for the distance on the plate between λ 5000 and λ 3900 is only 2 mm. The line at G is very broad and it is not impossible that $H\gamma$ may help in its formation.

It might be thought that the three stars, γ, δ, and λ *Sagittarii*, which are south and east of the star cloud under consideration and are of the K-type, have an injurious effect, but that this is not the case is shown by measures of the brightness of this region with a photometer of the type used by Yntema and described in *Groningen Publications*, No. 22. A provisional reduction of these measures gives the brightness of a square degree of the Milky Way in this very bright region equal approximately to 0.4 the light of a standard first magnitude star. The total intensity of the light of the three stars is equal to half that of a star of the first magnitude. The

angular aperture of the collimator lens may be taken as 3°. Accordingly the area of sky affecting the plate at any time was about seven square degrees. Then, assuming the most unfavorable condition possible, namely, concentrating the light from the three stars into one at the center of the region, this would amount to less than one-fifth of the total light falling on the plate at any time. As a matter of fact, these stars could send no light whatever to the plate unless the spectrograph was not moved at sufficiently short intervals of time. This occurred a few times, as it was necessary to carry on this exposure at the same time the writer was observing with the 60-inch reflector, and time could not always be taken to make the required changes. Had the exposure been made, either by having the spectrograph moved by a driving-clock or by continuous motion of the alt-azimuth mounting by the observer, the light of the three stars would never have reached the plate.

This result was of sufficient interest to continue the work, the same spectrograph again being loaned by the Lick Observatory. The next plate was exposed a total of 67^h52^m during May and June 1912, on the bright region at $\alpha = 18^h42^m$, $\delta = -8°$. This time the spectrograph was attached to the coelostat of the Snow telescope, a new set of gears providing the required speed of rotation. The plate is very much like the previous one, except that it is not quite so dense and the absorption line at 421 $\mu\mu$ is absent.

A third exposure was made during July and August 1912, on the region between β and η *Cygni*. This exposure amounted to 74^h11^m. The spectrograph was again attached to the coelostat of the Snow telescope.

The intensity of the last plate is about the same as that of the stronger plate of the *Sagittarius* region and the spectrum is similar in its general appearance, but the absorption line at 411 $\mu\mu$ is relatively stronger and the possible bright line at 416 $\mu\mu$ is relatively weaker.

The four plates taken of three bright portions of the Milky Way are therefore in agreement in indicating that *the integrated spectrum of the Milky Way is approximately of solar type.*

This result receives an independent check from a series of plates taken to test the possible selective absorption, or loss, of light in

space. On the strength of the Harvard result that the greater portion of the Milky Way stars are of the A-type, and assuming that this result could be extended to the fainter stars, it appeared possible to test the hypothesis that the more distant, and therefore fainter, stars are redder than the nearer brighter ones.

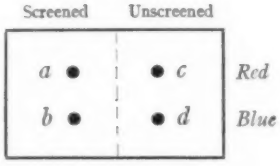
The regions selected for trial were centered at

$18^h30^m+10°$	$18^h30^m+5°$
18 40 +10	18 40 +5
18 50 +10	

and were photographed with the 60-inch reflector on red-sensitive plates.

On each plate two exposures of a region were made, one through a red and the other through a blue ray filter, the plate being moved a little in declination to separate the images of the two exposures. In addition one-half of the plate was kept covered by the slide of the plate-holder except during 30^s at the close of each 6^m of the exposure. Thus on the covered or "screened" half of the plate only the brighter stars are found. Accordingly it is possible to compare faint stars from the uncovered portion of the plate with bright stars from the screened portion.

The theory of the method, which was suggested by Professor Kapteyn is as follows: Let us consider two stars, one from each half of the plate, at the same distance from the optical axis so that there will be no differential aberration effect, and with the red images approximately equal. Though these images are of the same size, that for the screened portion of the plate will be of a star approximately two magnitudes brighter, and hence, on the average, a less distant star, than that whose image lies within the unscreened portion.



On the assumption that there is a greater absorption and scattering of blue than of red light in interstellar space, the screened

blue image b will be larger than the blue image d. Since all of the images compared are approximately of the same size the result will be free from photographic effects that might otherwise enter.

Let us put

$$a-c=m \tag{1}$$

$$b-d=n \tag{2}$$

On the basis of our assumption the value $n-m$ will be positive, that is

$$b-d>a-c$$

Subtracting (1) from (2)

$$(b-d)-(a-c)=n-m=\text{redness}=r$$
$$r=(b-a)+(c-d)$$

If r is positive the more distant stars are redder than those nearer at hand. Should r prove to be negative then the more distant stars are bluer.

The plates taken were measured in the following manner. A circle of 38 mm radius was drawn with its center at the center of the plate. No stars outside the circle were used on account of the large aberration beyond this limit. A star pair from the screened portion was then compared with any pair in the unscreened region that was approximately at the same distance from the center, the red images being as nearly as possible of the same size.

It will be unnecessary to go farther into the details of this matter. The result of the comparison of 76 pairs of stars on the five plates showed that, in the mean, the fainter stars are distinctly redder than the brighter ones. Had we been certain that the stars were all of approximately the same spectral type we should have strong evidence that the light from the fainter stars lost more of its blue than of its red component in traversing interstellar space, as compared with the brighter stars. The spectrograms, however, offer another explanation, namely, the ratio of A-type stars to other types decreases as the stars become fainter, and, from a certain undetermined magnitude on, the stars of approximately solar type predominate, so that the fainter stars, which give us the most of the Milky Way light, are in reality redder than the brighter

ones. These two investigations are therefore in agreement, although the original purpose of the second is not served.

It is unfortunate that the regions selected for the second investigation were not identical with those of which spectrograms were obtained, but the former plates were taken first. It was not advisable to try a spectrographic exposure on these regions as they are too faint, the exposure required being probably in the neighborhood of 200 hours. The first spectrogram was obtained at about the same time as the other series of plates. The results of the latter were therefore not published in the form originally intended.

The Harvard investigation probably contains very few, if any, stars fainter than the 8th magnitude. Both the investigations discussed here deal almost exclusively with stars fainter than this. It would be a matter of interest to know at what magnitude the ratio of A-type stars to all other types reaches a maximum, as well as the magnitude at which the other types begin to predominate. From the material at hand, however, there seems to be no satisfactory method of obtaining these values.

In regard to the spectrographic exposures on the Milky Way it should be stated that every precaution was taken to prevent sunlight from any source reaching the plate. The exposures were not begun until the sky was thoroughly dark and were closed before dawn. Furthermore, no exposure was made while the moon was above the horizon.

In conclusion, the writer desires to express his great appreciation to Director Campbell of the Lick Observatory for the use of the spectrograph, and to Mr. P. J. Van Rhyn, who assisted in the exposure of the last plate.

Note added October 22.—The reproductions of the spectra are not very satisfactory owing to the enlargement required and the very coarse grain of the original negatives. In *a* the G-group is faintly indicated, H and K form a broad band, and the shadings to the left are the ultra-violet bands referred to above.

MOUNT WILSON SOLAR OBSERVATORY
August 27, 1912

THE *ALGOL* VARIABLE *RR DRACONIS*[1]

By FREDERICK H. SEARES

Among the variable stars included in the program of observations carried on at the Laws Observatory was the object discovered by Madame Ceraski in 1904 which has since received the designation *RR Draconis*. When first placed on the observing-list the character of the variation was unknown, but the measures of the first two nights showed the star to be an *Algol* of unusual interest on account of the range and rapidity of its variation. It was found normally to be a little brighter than the tenth magnitude, but during the eclipses, which occur at intervals of two days and twenty hours, it became so faint as to disappear from view in the $7\frac{1}{2}$-inch equatorial with which the observations were made. The brightness at the time of disappearance was apparently a little less than the thirteenth magnitude, so that the range covered by the visual observations was in excess of three magnitudes, the average change being at the rate of nearly a magnitude per hour.

Though the nature of the variation during the two hours that the star remained invisible in the Laws Observatory equatorial could not be determined, it was possible from the steepness of the curve to derive the epochs of minima, and consequently the period, with precision. The photometer used was of the equalizing wedge type designed by Pickering,[2] and as the wedge itself had been carefully calibrated and tested[3] it was possible to reduce the measures to an absolute scale of magnitudes, which were then combined to form a mean light-curve. The details of the observations, which extended from June 29, 1905, to September 25, 1906, and their discussion were published in *Laws Observatory Bulletin*, No. 9. The elements found were

$$\text{Min.} = \text{J.D. } 2417026.682 + 2^{d}.831079\ E \quad \text{G.M.T.}$$

with an estimated uncertainty in the period of 1^{s}.

[1] *Contributions from the Mount Wilson Solar Observatory*, No. 64.

[2] *Astrophysical Journal*, **13**, 249, 1901.

[3] *Laws Observatory Bulletin*, No. 7, 1905.

The light-curve derived from the visual observations, reduced to the zero point of the Harvard Polar Sequence in a manner to be described later, is given in Table I. The phase values in the first and third columns are counted from minimum. With the exception of a remark by Hartwig,[1] indicating a correction of $+20^m$ to the ephemeris derived from the above elements for 1911, no further results seem to have been published.[2]

TABLE I

VISUAL LIGHT-CURVE—LAWS OBSERVATORY

Phase	Mag.	Phase	Mag.
0^d035	12.96	0^d100	10.51
0.040	12.58	0.110	10.35
0.045	12.29	0.120	10.23
0.050	12.02	0.130	10.13
0.055	11.79	0.140	10.03
0.060	11.59	0.150	9.95
0.065	11.40	0.160	9.88
0.070	11.23	0.170	9.82
0.075	11.08	0.180	9.77
0.080	10.94	0.190	9.74
0.085	10.81	0.200	9.72
0.090	10.70	0.210	9.71
0.095	10.60	0.220	9.70
0.100	10.51	0.230	9.70

In order to determine the total range and the character of the variation for the part of the curve not included in Table I, the star was followed photographically throughout the greater part of an eclipse on August 7, 1912, with the 60-inch reflector of the Mount Wilson Solar Observatory.

Eight plates including 61 exposures of 2^m each and 4 of 1^m, covering a period of six hours, were obtained; and on the following night three additional plates for the determination of the normal brightness of the variable and the zero point of the magnitude scale were made. The observational details are contained in Table II. On four of the plates exposures were made with various

[1] *Vierteljahrsschrift der Astronomischen Gesellschaft*, **46**, 230, 1911. This correction and the ephemeris given by Hartwig are based on the provisional elements in *Laws Observatory Bulletin*, No. 9, p. 135. The final elements on p. 138 are more precise.

[2] The observations by Lehnert, *Astronomische Nachrichten*, No. 4596, arrived just as this paper was sent to the printer.

combinations of apertures for the purpose of establishing the magnitude scale of the comparison stars. For Nos. 784 and 785 the exposures with the full aperture alternated with those made with 32- and 14-inch diaphragms; for 790 the order of apertures was 60, 32, 14, 14, 32, and 60 inches; and for 791, 32, 9, 9, and 32 inches. Plate 792 was exposed both to the field of the variable and to the Pole, the order being, variable, Pole, Pole, variable.

TABLE II

LIST OF PLATES

Date	G.M.T.	Plate No.	Exp. Time	No. Exp.	Aperture	Seeing
1912						
August 7....	$16^h\,55^m$–$17^h\,30^m$	778	2^m	9	60	5
	17 37 –18 16	779	2	8	60	4
	18 23 –19 1	780	2	8	60	3
	19 13 –19 47	781	2	9	60	3
	19 54 –20 29	782	2	9	60	2–3
	20 35 –21 11	783	2	9	60	2–3
	22 3 –22 37	784	2	9	60, 32	3
	22 42 –23 1	785	1	4	60, 14	3
August 8....	18 59 –19 11	790	1	6	60, 32, 14	5
	19 17 –19 30	791	2	4	32, 9	5
	19 34 –19 55	792	2	4	32	6

Fourteen comparison stars, all near the variable, were selected for use in the determination of the magnitude scale. The brightest of these, which is No. 1 of the list, is *B.D.*+62°1642, 9.3 mag. The others, all of which are fainter than the *Durchmusterung* limit, may be identified with the aid of the chart of the field shown in Fig. 1. On the diaphragm plates all of the comparison stars were measured. The resulting mean scale readings, corrected for distance error, are given in Table III. The subscripts in the headings of the table refer to the aperture. The mean scale readings of the stars of the Polar Sequence shown on plate 792 are also given in Table III, in the last column. The corresponding Polar Sequence numbers and magnitudes taken from *Harvard Circular*, No. 170, are in the two preceding columns. The scale readings for such stars as were used for the reduction of the remaining plates are given in Table V.

The data for plates 784–791 were reduced in the usual manner[1]

[1] A general account of the methods of observation and reduction followed in photometric work with the 60-inch reflector is in preparation and will be published shortly.

for the determination of the magnitudes of the comparison stars. The various scales were brought to an arbitrarily chosen common zero point, and the separate values for the brightness of each star combined to form the provisional magnitude given in the second column of Table IV. The five columns immediately following contain the deviations of the separate determinations from the mean provisional magnitudes. There is no evidence of appreciable

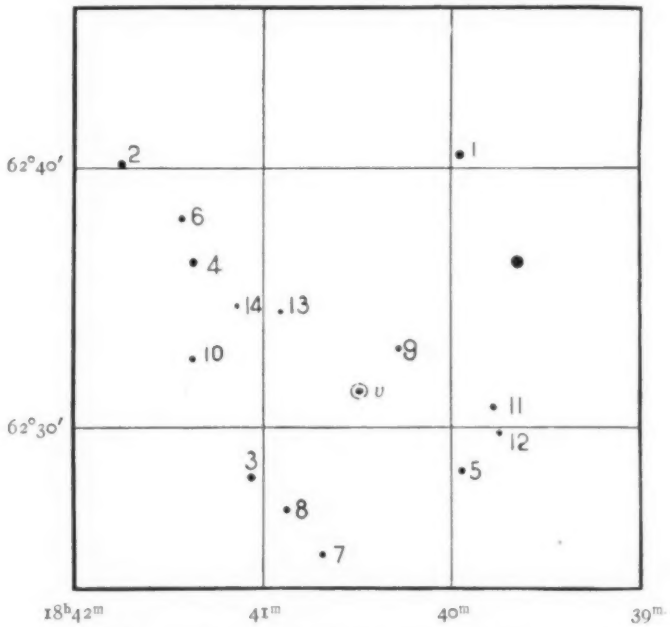


FIG. 1.—The comparison stars used with *RR Draconis*

systematic differences between the different determinations of the scale, and in general the consistency of the results is satisfactory. The average deviation of a single magnitude, including the effect of scale error over a range of four magnitudes is 0.096 mag., which corresponds to a probable error of ±0.081 mag.

To reduce the provisional magnitudes to the zero point of the Harvard Polar Sequence the values of the scale readings of the Polar Sequence stars given in Table III for plate 792 were plotted

against the corresponding Polar Sequence magnitudes. From the curve thus established were read the values of the magnitudes corresponding to the scale readings of the comparison stars photographed on the same plate. These are entered in the eighth column of Table IV. The differences between these values and the corresponding provisional magnitudes in the second column appear in the ninth column and, disregarding extinction, represent separate determinations of the constant for the reduction of the provisional magnitudes to the Polar Sequence zero point. The interval of brightness covered by the polar stars is too short to afford a reliable comparison of the scale derived for the comparison stars with that of the Polar Sequence.

TABLE III

MEAN SCALE READINGS FOR MAGNITUDE SCALE PLATES

Star	784		785		790			791		792			
											Polar Sequence		
	s_{60}	s_{32}	s_{60}	s_{24}	s_{60}	s_{32}	s_{24}	s_{32}	s_{9}	s_{32}	No.	Mag.	s_{32}
0 ..	3.2	8.4	5.0	12.6	2.9	8.8	11.8	6.8	12.8	6.1	14	10.52	9.8
1 ..	3.3	8.5	5.0	12.6	3.6	9.0	12.2	7.3	13.3	6.6	7_7	10.57	10.0
2 ..	.....	13.8	10.9	.....	.....	15.5	.....	14.2	.....	13.7	5_5	10.68	10.4
3 ..	9.3	14.3	11.4	.....	9.4	15.6	.....	14.1	.....	14.3	18	11.92	14.4
4 ..	9.2	14.3	11.8	.....	9.9	15.9	.....	14.5	.....	14.0	19	12.28	14.7
5 ..	9.3	14.6	12.2	.....	10.9	17.8	.....	14.6	.....	14.7	7_8	12.31	14.8
6 ..	10.4	15.8	12.2	.....	11.8	.....	.....	15.5	.....	16.3	20	12.59	16.6
7 ..	10.0	15.2	12.5	.....	11.4	17.9	.....	16.7	.....	15.5	11_7	12.87	18.5
8 ..	11.2	16.3	14.1	.....	11.9	.....	.....	17.4	.....	17.4			
9 ..	11.1	17.2	13.6	.....	12.0	.....	.....	17.6	.....	17.8			
10 ..	11.6	17.6	13.6	.....	12.1	.....	.....	17.4	.....	17.8			
11 ..	11.5	.....	13.9	.....	13.6	.....	.....	.....	.....	18.0			
12 ..	12.5	.....	14.8	.....	14.3	.....	.....	.....	.....	.....			
13 ..	13.6	.....	15.5	.....	14.7	.....	.....	.....	.....	.....			
14 ..	13.9	.....	15.4	.....	15.1	.....	.....	.....	.....	.....			

The zenith distance of the variable for the mean time of exposure of plate 792 was 34°.9. That of the Pole is 55°.8. The difference is greater than desirable, but could not be avoided without a serious disarrangement of the observing program. In the absence of special extinction tables for Mount Wilson an approximate value of 0.20 mag. has been derived from those of Müller.[1]

[1] *Photometrie der Gestirne*, p. 515, 1897.

The visual extinction has been doubled, and a rough interpolation has been made for elevation above sea-level. The application of the extinction correction to the mean of the quantities in the ninth column of Table IV gives for the reduction constant of the provisional magnitudes 8.95 mags. The resulting magnitudes adopted for the comparison stars are printed in the last column of Table IV.

TABLE IV

MAGNITUDE OF COMPARISON STARS

Star No.	Provisional Mag.	Residuals					Mag. from Polar Plate 792	Red. to Polar Sequence	Adopted Mag.
		784 60–32	785 60–14	790 60–32	790 60–14	791 32–9			
1.....	0.90	+ 1	+17	−11	−13	+ 6	.....	.....	9.85
2.....	3.03	+ 5	+ 1	+ 2	.....	− 7	11.87	8.84	11.98
3.....	3.12	−12	−10	+ 8	+11	+ 5	12.07	8.95	12.07
4.....	3.23	+ 1	−14	+ 5	+ 5	+ 3	11.98	8.75	12.18
5.....	3.40	+10	+ 7	−23	−12	+17	12.20	8.80	12.35
6.....	3.68	− 3	+15	−16	−14	+16	12.53	8.85	12.63
7.....	3.70	+15	+ 5	− 2	+ 2	−21	12.38	8.68	12.65
8.....	4.02	+ 8	−25	+15	+17	−12	12.70	8.68	12.97
9.....	4.03	− 5	− 5	+13	+14	−18	12.76	8.73	12.98
10.....	4.06	−18	− 2	+13	+14	− 8	12.76	8.70	13.01
11.....	4.28	+16	+ 9	−11	−15	.....	12.80	8.52	13.23
12.....	4.58	+10	+ 3	− 2	− 9	.....	.....	.....	13.53
13.....	4.80	− 7	− 2	+ 7	0	.....	.....	.....	13.75
14.....	4.89	− 9	+11	+ 4	− 5	.....	.....	.....	13.84
Means		0.086	0.090	0.094	0.101	0.113	.....	8.75	
Extinction.....								0.20	
								8.95	

With the magnitudes of the comparison stars once established, the determination of the brightness of the variable becomes a simple matter. It is only necessary to plot the scale readings of the comparison stars for any plate against the adopted magnitudes and then read from the resulting curve the magnitude corresponding to the scale reading of the variable.

The scale readings for the comparison stars derived from plates 778–783 and given in Table V are the means of the eight or nine exposures on each of these plates, a single estimate with the photometric scale being made for each image. The results for the

variable from these plates are given separately for each image in Table VI. These are the means of three estimates with the scale.

Since the mean image was used for the comparison stars and separate images for the variable, it was necessary to correct the scale readings of the variable for the systematic deviations of the various exposures from the mean of them all. The corrections were derived by subtracting for each star the scale reading of each

TABLE V

MEAN SCALE READINGS OF COMPARISON STARS

Star No.	Mag.	778	779	780	781	782	783
1.....	9.85	4.1	.....	.....	.....	3.9	3.7
2.....	11.98	9.2	.....	.....	.....	8.8	8.0
3.....	12.07	10.6	.....	.....	.....	9.7	9.3
4.....	12.18	10.4	.....	.....	10.7	11.0	10.7
5.....	12.35	10.3	.....	.....	10.3	10.0	9.4
6.....	12.63	11.0	.....	.....	11.6	.....	.....
7.....	12.65	11.3	.....	.....	11.1	.....	.....
8.....	12.97	12.0	12.1	11.8	12.0	.....	.....
9.....	12.98	11.7	12.0	11.7	12.0	.....	.....
10.....	13.01	12.3	12.7	12.3	12.3	.....	.....
11.....	13.23	12.0	12.1	11.8	11.9	.....	.....
12.....	13.53	.....	13.0	12.7	12.9	.....	.....
13.....	13.75	.....	14.2	14.2	14.2	.....	.....
14.....	13.84	.....	14.1	14.1	14.1	.....	.....

image from the mean scale reading for all the images of the star. The average of the differences for the same exposure for all the stars gives the correction which will reduce the scale readings for that exposure to the mean exposure for the plate. The results for the various exposures on the different plates are given in Table VII, the unit being 0.1 of a scale interval.

The corrections are in all cases small and show that the atmospheric conditions, both as to transparency and steadiness, were satisfactorily constant. The mean values in the last column give a pretty clear indication that the plates are free from the photographic effect which sometimes causes the first of a series of exposures on a plate to be systematically brighter than the last.[1]

[1] Professor Pickering kindly allows me to quote from a personal letter as follows: "When several exposures are made on a plate, the first and last being short and equal, the images formed by the first exposure are almost always brighter than those formed by the last. The average difference is about a quarter of a magnitude."

The scale readings of the variable in Table VI are the corrected values. In the case of plates 784 and 785 the rate of change in the variable is so small that the mean scale reading of the various exposures is used, the only distinction being a separation of the

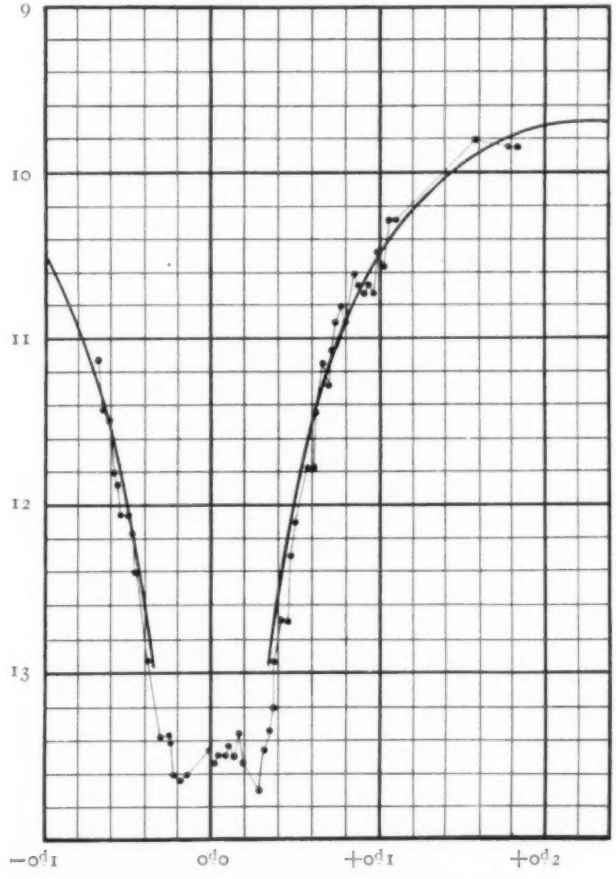


FIG. 2.—Light-curve of *RR Draconis* at eclipse

Broken line—Photographic observations
Full line—Laws Observatory visual light-curve

results for the different apertures. The seventh column of Table VI contains the values of the brightness of the variable read from the magnitude curves for the various plates. The results are also shown graphically in Fig. 2.

TABLE VI

MEAN SCALE READINGS AND MAGNITUDES OF *RR Draconis*

1912, August 7 and 8

Plate	Exposure	G.M.T.		Phase	s	Mag.	Phot.−Vis.
778.....	1	$16^h\ 56^m$	$0^d.705$	$-0^d.068$	7.3	11.12	−18
	2	17 1	0.709	−0.064	8.1	11.43	− 1
	3	5	0.712	−0.061	8.2	11.49	− 6
	4	9	0.714	−0.059	9.0	11.80	+16
	5	13	0.717	−0.056	9.2	11.88	+14
	6	17	0.720	−0.053	9.6	12.05	+17
	7	21	0.723	−0.050	9.6	12.05	+ 3
	8	25	0.726	−0.047	9.9	12.16	− 2
	9	29	0.728	−0.045	10.5	12.40	+11
779.....	1	17 38	0.735	−0.038	11.8	12.91	+18
	2	51	0.744	−0.029	13.0	13.37	−13*
	3	55	0.746	−0.027	13.0	13.37	−13
	4	59	0.749	−0.024	13.1	13.41	− 9
	5	18 3	0.752	−0.021	13.6	13.60	+10
	6	7	0.755	−0.018	13.7	13.63	+13
	7	11	0.758	−0.015	13.6	13.60	+10
	8	15	0.760	−0.013	13.6	13.60	+10
780.....	1	18 32	0.772	−0.001	13.1	13.46	− 4
	2	36	0.775	+0.002	13.3	13.53	+ 3
	3	40	0.778	+0.005	13.2	13.49	− 1
	4	44	0.781	+0.008	13.2	13.49	− 1
	5	48	0.784	+0.011	13.0	13.43	− 7
	6	52	0.786	+0.013	13.2	12.49	− 1
	7	56	0.789	+0.016	12.8	13.35	−15
	8	19 0	0.792	+0.019	13.3	13.53	+ 3
781.....	1	19 14	0.802	+0.029	13.8	13.70	+20
	2	18	0.805	+0.032	13.1	13.45	.....
	3	22	0.808	+0.035	12.8	13.33	+37
	4	26	0.810	+0.037	12.5	13.20	+39
	5	30	0.812	+0.039	11.9	12.93	+27
	6	34	0.815	+0.042	11.4	12.69	+23
	7	38	0.818	+0.045	11.4	12.69	+40
	8	42	0.820	+0.047	10.5	12.30	+12
	9	46	0.823	+0.050	10.1	12.10	+ 8
782.....	1	19 55	0.830	+0.057	8.9	11.78	+ 7
	2	59	0.832	+0.059	8.9	11.78	+15
	3	20 4	0.836	+0.063	8.0	11.44	− 4
	4	8	0.839	+0.066	7.2	11.14	−23
	5	12	0.842	+0.069	7.6	11.29	+ 3
	6	16	0.844	+0.071	7.0	11.06	−14
	7	20	0.847	+0.074	6.6	10.90	−21
	8	24	0.850	+0.077	6.4	10.81	−21
	9	28	0.852	+0.079	6.6	10.90	− 7
783.....	1	20 36	0.858	+0.085	6.6	11.00	+19
	2	40	0.861	+0.088	5.8	10.68	− 6

* The residuals for the phase values $-0^d.03$ to $+0^d.03$ refer to the minimum *photographic* brightness of 13.50 mags.

TABLE VI—*Continued*

Plate	Exposure	G.M.T.		Phase	s	Mag.	Phot.—Vis.
783.....	3	20^h44^m	0^d864	$+0^d091$	5.9	10.72	+ 4
	4	48	0.867	+0.094	5.8	10.68	+ 6
	5	52	0.870	+0.097	5.9	10.72	+16
	6	56	0.872	+0.099	5.3	10.48	− 5
	7	21 0	0.875	+0.102	5.5	10.56	+ 8
	8	6	0.879	+0.106	4.8	10.28	−13
	9	10	0.882	+0.109	4.8	10.28	− 9
784.....	60	22 20	0.931	+0.158	3.2	9.81	− 8
	32	20	0.931	+0.158	8.4	9.81	− 8
785.....	60	22 51	0.952	+0.179	5.0	9.85	+ 7
	14	54	0.954	+0.181	12.6	9.85	+ 8
790.....	60	19 5	1.796	+1.023	2.9	9.61	− 9
	32	5	1.796	+1.023	8.8	9.78	+ 8
	14	5	1.796	+1.023	11.8	9.71	+ 1
791.....	32	19 23	1.808	+1.035	6.8	9.69	− 1
	9	23	1.808	+1.035	12.8	9.69	− 1
792.....	32	19 44	1.822	+1.049	6.1	9.70	0

A curious fact is brought to light by the magnitude curves for plates 778–783 drawn from the data in Table V, namely, that although the adopted magnitudes afford a consistent representation of the measures of plates 784–791 as shown by the residuals in Table IV, they do not represent satisfactorily the scale readings for plates 778–783. This is clearly shown by the residuals in Table VIII which are the deviations for these plates of the adopted magnitudes from those corresponding to the observed scale readings, the latter being read directly from the curves. These residuals are clearly systematic, not in the sense that would indicate an error of scale, but as though the comparison star magnitudes were affected with large accidental errors. Most of the stars, for example, show deviations of the same sign and of the same order of magnitude for all of the plates of this group upon which they were measured. The difficulty is not with the comparison stars, however, but with the plates of this particular group for which the error depending on the distances of the stars from the axis of the instrument was undoubtedly abnormal. It has been found that comparatively small temperature deformations of the mirror are sufficient to

modify radically the character and amount of the correction required to compensate this error. The mirror is known to have been in an abnormal condition during the early part of the night when the plates in question were exposed, but as it is impossible to determine the distance error for each plate separately, the correction for normal figure was applied. The systematic deviations in Table VIII represent the uncertainty of such a method of procedure. The comparison star magnitudes were derived from plates made after the mirror had become normal, or nearly so. That they are not also affected by an uncertainty in the distance error is shown by forming a similar table of deviations for the results obtained with the 32-inch diaphragm on plates 784, 790, 791, and 792. With this aperture the distance error is vanishingly small for the extent of field covered. The comparison of the adopted magnitudes with the scale readings for the diaphragm exposures should therefore reveal any uncertainty that may have entered through the full aperture exposures on the plates from which the magnitudes were derived. The resulting deviations are wholly accidental, however, from which we may conclude that the comparison star magnitudes are unaffected by a residual distance error.

The whole question of the determination of the proper correction to be applied for distance error is one that has given much trouble in photometric work with the 60-inch reflector. With care, however, the difficulties can largely be avoided, as will appear from a detailed discussion to be published in a later paper.

In the present case the matter is complicated by the fact that there is available but a single bright comparison star. Owing to the absence of other stars of similar brightness the magnitude curves were naturally drawn *through* the points corresponding to the bright star. For this reason no residuals appear in Table VIII for this object. The brighter magnitudes of the variables may therefore be slightly in error in such a way as to affect systematically the results for a plate; but only plates 782 and 783 can be so affected, and the smoothness of the resulting light-curve indicates that the errors must be small. The images of the variable itself are unaffected, as they were always near the axis. The results for normal light are also unaffected by residual distance error, for they were

derived from plates made when the figure of the mirror was satisfactory, and depend almost exclusively upon exposures with diaphragms of 32, 14, and 9 inches for all of which the distance correction is negligible.

TABLE VII

REDUCTION TO MEAN EXPOSURE

Unit = 0.1 Scale Interval = 0.35 Mag.

EXPOSURE	778	779	780	781	782	783	784	MEANS	
								s	Mag.
1.....	0	+2	...	+1	−4	+1	0	0.00	0.00
2.....	+1	...	+1	+1	−2	+3	+1	+0.08	+0.03
3.....	−1	−3	+3	+2	−3	+1	+3	+0.03	+0.01
4.....	+2	−2	+4	−1	−1	+3	0	+0.07	+0.02
5.....	+2	−1	−1	−1	+2	−7	−2	−0.11	−0.04
6.....	+2	0	−1	−3	+1	+2	0	+0.01	0.00
7.....	−1	+1	−2	+1	+2	−4	−1	−0.06	−0.02
8.....	−1	+1	−1	−3	0	+3	−1	−0.03	−0.01
9.....	−3	+1	−1	+2	+3	0	−2	0.00	0.00

TABLE VIII

COMPARISON STAR RESIDUALS

Star No.	778	779	780	781	782	783
1.....	....	....	....	....	....	....
2.....	−10	....	....	....	−23	−43
3.....	+36	....	....	....	0	0
4.....	+18	....	....	+20	+50	+43
5.....	− 2	....	....	−17	−15	−24
6.....	0	....	....	+17	....	....
7.....	+ 5	....	....	−12	....	....
8.....	+ 3	+ 6	+ 6	0	....	....
9.....	−10	+ 1	0	− 1	....	....
10.....	+13	+25	+19	+10	....	....
11.....	−22	−20	−20	−31	....	....
12.....	....	−16	−20	−16	....	....
13.....	....	+ 7	+ 6	+ 8	....	....
14.....	....	− 7	− 7	− 4	....	....

From a comparison of the times corresponding to a series of points on the steepest parts of the two branches of the light-curve the observed time of minimum is found to be $18^h\ 33^m$ G.M.T. As the reduction to the sun is $+1^m,^1$ the observed heliocentric mini-

[1] *Laws Observatory Bulletin*, No. 9, 135, 1907.

mum is 1912, August 7, $18^h\ 34^m$ G.M.T. with an uncertainty of 1^m. The minimum calculated from the original elements given above is $18^h\ 45^m.2$. The correction to the ephemeris is therefore $-11^m.2$. As 917 periods separate the present minimum from the epoch of the elements, the correction to the period is $-0^s.74$ or $-0^d.000009$. The revised elements are therefore

$$\text{Min.} = \text{J.D. } 2417026.682 + 2^d.831070\ E \quad \text{G.M.T.}$$

with an uncertainty in the period of $0^s.07$.

From the last three plates listed in Table VI, all of which were exposed during normal light, the maximum brightness of the variable was found to be 9.70 mags. The result of the Laws Observatory visual observations, referred to the system of the Potsdam *Durchmusterung*, was 9.98 mags.,[1] which was obtained by a reference to stars 9950 and 10372 of the *Generalkatalog* whose colors are respectively GW and WG−. The spectrum of the variable is apparently of the A-type;[2] and though its exact classification is doubtful, there can be no question but that it belongs among the white stars. It is not likely therefore that the reference of the variable to the *P.D.* stars is greatly affected by color difference. The mean reduction of the two Potsdam stars to the system of the Harvard Observatory is −0.22 mag.[3] The normal brightness derived from the visual observations, referred to the Harvard system, is therefore 9.76 mags. The determination of this quantity depends upon the measurement with the equalizing wedge photometer of intervals of 2.9 and 2.4 mags.,[4] and the close agreement of the visual and photographic results indicates a satisfactory accordance of the visual and photographic magnitude scales. Further data bearing on this point will appear later.

The phase values in the fifth column of Table VI were obtained by subtracting the observed time of geocentric minimum from the epochs of observation in column four. Assuming that the interval of constant minimum light falls within the phase limits $\pm 0^d.030$,

[1] *Op. cit.*, p. 128.

[2] *Annals Harvard College Observatory*, **56**, 189.

[3] *Publikationen des Astrophysikalischen Observatoriums zu Potsdam*, **17**, XXXIV, 1907.

[4] *Laws Observatory Bulletin*, No. 9, 128, 1907.

there are available for the determination of the minimum brightness sixteen observations. The mean result is 13.50 mags. The inclusion of the first three observations is questionable, though the symmetry of the curve indicates that they fall within the period of constant light. If excluded, the minimum brightness becomes 13.53 mags.

The ordinate of the visual light-curve for normal light, referred to the star *B.D.*+62°1639, is +0.34 mag.[1] As the normal brightness has been found to be 9.70 mags., the visual curve may be reduced to the zero point of the Polar Sequence by adding to its ordinates the constant 9.36 mags. The values given in Table I were derived in this manner, and the corresponding curve is shown with the photographic observations in Fig. 2. The values of the deviations in hundredths of a magnitude appear in the last column of Table VI. Within the phase limits $\pm 0^{d}.030$ for which the visual curve is lacking, the residuals are referred to the minimum photographic brightness.

It will be noted that for two magnitudes below normal light the photographic variation is closely represented by the visual light-curve. From this point on, the photographic variation is greater than the visual, the divergence amounting to about three-tenths of a magnitude in an interval of a magnitude and a half. If it be assumed that the photographic and visual range of variation are necessarily the same, this would imply a relative error in the photographic and visual scales of 20 per cent; but in view of their close agreement for the first two magnitudes of the light-change and for the intervals measured in determining the normal brightness referred to above this is unlikely, and the evidence, as far as it goes, indicates that the photographic range is really greater than the visual. For the greater part of the curve the variation is rapid, and the last magnitude of the change requires but twenty minutes.

The value of the range of variation given by the photographic observations is 3.80 mags. As to the reliability of this result, the following may be remarked: The uncertainty of the values for both maximum and minimum brightness of the variable as indi-

[1] *Op. cit.*, p. 136.

cated by the deviations of the individual values from the means is only 0.02 mag. The actual uncertainty for maximum light is considerably in excess of this, however, owing to the fact that there is but one comparison star at all comparable in brightness with the variable. Any error in the brightness of this star will therefore enter practically to its full value into the calculated brightness of the variable. The comparison star residuals in Table IV indicate that the uncertainty for the object in question, No. 1, is ±0.04 mag., which includes the effect of scale error, but not that of the zero point. For minimum light on the other hand the course of the magnitude curves from which the brightness of the variable was read is determined by several comparison stars, so that, disregarding again the error of the zero point, the uncertainty entering through the comparison stars into the minimum brightness is comparatively small. From a consideration of these points it seems likely that the observed range in brightness is correct within a tenth of a magnitude, unless there is an unsuspected error affecting similarly all of the separate determinations of the magnitude scale.

So far as the amount of variation is concerned *RR Draconis* stands third among the 134 variables listed in *Harvard Annals*, **56**, 187. It is exceeded only by *RW Tauri*, for which the range is 3.9 mags., and by *RZ Aurigae*, to which a photographic amplitude of 5.0 mags. is assigned, though the visual variation is much less, the results by various observers being 2.1[1] and 1.7 mags.[2]

Though it is not at present desirable to enter upon a detailed investigation of the system, certain points may be noted. The observed eclipse is, of course, that of a bright object by one relatively faint, for such is always the character of the eclipse corresponding to a principal minimum. Moreover, the data are sufficient to show that the eclipse is total and not annular—that the darker star is the larger of the two.

If λ_p and λ_s represent the light received during the principal and secondary minima, respectively, and if k be the ratio of the

[1] Hartwig, *Astronomische Nachrichten*, **178**, 395, 1908.

[2] Pračka, *ibid.*, **185**, 291, 1910.

diameter of the smaller star to that of the larger, we have[1] when the principal minimum is the total eclipse,

$$\lambda_s = 1 - k^2\lambda_p \tag{1}$$

and when the principal minimum is the annular eclipse

$$\lambda_s = \frac{1-\lambda_p}{k^2} \tag{2}$$

Corresponding to the range 3.80 mags. $\lambda_p = 0.03$, whence (1) and (2) become, respectively,

$$\lambda_s = 1 - 0.03k^2 \text{ (Prin. Min. is Total Ecl.)} \tag{3}$$

$$\lambda_s = \frac{0.97}{k^2} \text{ (Prin. Min. is Annular Ecl.)} \tag{4}$$

subject to the condition $\lambda_s < 1$. Equation (3) is obviously possible, but in order that (4) may exist we must have $k^2 > 0.97$.

From the general relations for a circular orbit of inclination i

$$\frac{(1-k)^2}{(1+k)^2} = \frac{\cot^2 i + \sin^2 \theta_2}{\cot^2 i + \sin^2 \theta_1} \tag{5}$$

in which θ_1 and θ_2 are the longitudes of the smaller star, counted from minimum, corresponding to the first and second contacts of eclipse, that is, to the beginning of the variation and of the constant minimum phase. Since i is necessarily nearly 90° we may write (5) in the form

$$\frac{1-k}{1+k} = \frac{\sin \theta_2}{\sin \theta_1} \tag{6}$$

and derive an approximation for k from

$$k = \frac{\sin \theta_1 - \sin \theta_2}{\sin \theta_1 + \sin \theta_2} \tag{7}$$

This value will in general be too large; for since $\sin \theta_1 > \sin \theta_2$, the true value of $(1-k)/(1+k)$ from (5) will be larger than that given by (6), and hence the true k will be *less* than the approximation from (7). If therefore the value of k from (7) is such that $k^2 < 0.97$, there is no possibility of the principal minimum corresponding to annular eclipse. Such in fact is the case, for the phase values of

[1] Russell, "On the Determination of the Orbital Elements of Eclipsing Variable Stars," *Astrophysical Journal*, **35**, 323, 1912.

the first and second contacts are $0^d.22$ and $0^d.03$, respectively. From these we find from (7) for the maximum value of k, 0.76, and of k^2, 0.58, which is far below the limit corresponding to the association of the annular eclipse with the principal minimum. The estimation of the times of contact is naturally very uncertain, but even if we assume such extreme values for the phases as $0^d.30$ and $0^d.015$ the result for the maximum k^2 is 0.81, which is still safely below the limit. Such phase values are, however, clearly excluded by the observational data, and we can only conclude that the eclipse corresponding to the principal minimum is total. The system is therefore remarkable in that the smaller of the two stars is the brighter.

This result strengthens the conclusion stated above as to the probability of the photographic range being in excess of the visual. In fact with such a system the photographic range must almost certainly be greater than the visual, for, since the fainter body is the larger, the chances are very strongly in favor of its being redder than the brighter. During normal brightness the light is preponderatingly that of the brighter star, which is of the A-type, and the photographic and visual magnitudes will be sensibly the same. During the principal minima the A-type star is wholly eclipsed by its larger and redder companion. As the latter is fainter photographically than visually, the photographic range of variation must be greater than the visual. Moreover, for a large part of the light-curve there will be no appreciable difference between the photographic and the visual magnitudes, for the relatively small amount of light emitted by the darker star will not form any considerable portion of the total received until the eclipse is well under way.

Finally it may be noted that equation (3) shows that the light during the secondary minimum can scarcely be less than 0.98, which corresponds to an amplitude of only 0.02 mag.

MOUNT WILSON SOLAR OBSERVATORY
September 6, 1912

ON DARKENING AT THE LIMB IN ECLIPSING VARIABLES. II

By HENRY NORRIS RUSSELL and HARLOW SHAPLEY

3. *Annular and partial eclipses.*—The discussion of annular and partial eclipses of stars whose disks appear to be darkened toward the limb has involved more labor in computation than any other part of the theory of eclipsing variables. The results may, however, be given in relatively small compass, and the actual reduction of the observations of any star, with the aid of the tables here published, is almost as simple as on the assumption of disks of uniform brightness.

The light-changes due to the eclipse—partial or annular—of a disk completely darkened toward the edge by a smaller one have been computed by the method previously discussed,[1] supplemented by other graphical methods of integration. The percentage of light obscured increases steadily during the partial phases; and after the second contact it continues to increase, as the eclipsing disk moves toward the brighter central region of the other, so that computations had to be made for the annular as well as for the partial phases. On plotting the resulting light-curves, it was at once evident that there was no discontinuity in the curve on passing from the partial to the annular phases. This is illustrated by the curves in Fig. 1, which are drawn accurately to scale, and which show the variations in stellar magnitude due to central annular eclipses of a star, completely darkened toward the limb according to the "cosine law" already assumed, by a smaller one of one-ninth the brightness and k times the radius. The points on the curves corresponding to the beginning and end of the annular phase are indicated by short vertical lines. It is clear that there can be no hope at all of determining the duration of the annular phase from the observed curve. Such a light-curve, in fact, would undoubtedly be taken as evidence that the eclipse was partial. The small dots

[1] *Astrophysical Journal*, **36**, 239, 1912.

on one-half of the light-curve represent points on curves computed for partial eclipses of uniform disks, with the same depth of both primary and secondary minima, and so determined as to lie as close as possible to the curves previously computed. The agreement of points and curve is as good as might be expected from the best observations, the only sensible difference being that the "uniform" brightness gives, as usual, a shorter duration for the extreme phases near first contact. It is of interest to compare the elements

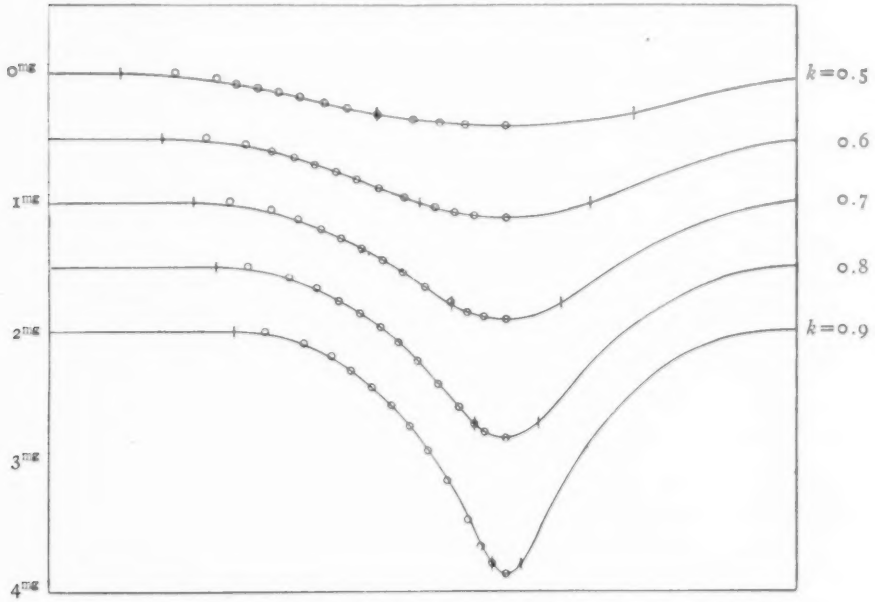


FIG. 1.—Light-curves due to annular eclipses of a star darkened toward the limb

derived from these light-curves on one hypothesis with those used in computing them on the other, as is done in the accompanying table. While the larger of the two "darkened" stars is the brighter, the smaller of the two "uniform" disks which give a similar light-curve is the more luminous. It is easy to show that this will always be true. Let L_1 be the light of the larger of the two "darkened" stars, and $1-L_1$ that of the smaller, and let k_d be the ratio of their radii. The loss of light at the total eclipse is $1-L_1$, and at central annular eclipse βL_1, where $\beta=1-(1-k_d^2)^{\frac{1}{2}}$. Unless $1-L_1$ is small,

the existence of a constant phase at the corresponding minimum will be detected. The case at present under discussion can only arise when the eclipse at principal minimum is annular. If we try to represent these minima by partial eclipses of magnitude a_0 of uniform disks, the ratio of whose radii is k_u, we must have, if we assume that the smaller star undergoes eclipse at principal minimum, $k_u^2 a_0 = k_u^2 \beta L_1 + (1 - L_1)$; on the opposite hypothesis, $k_u^2 a_0 = \beta L_1 + k_u^2 (1 - L_1)$. Now k_u^2, a_0, and L_1 must all be positive and less than

Elements Assumed for "Darkened" Stars					Elements Derived from the Light-Curve Assuming "Uniform" Disks						
k_d	r_{1d}	r_{2d}	L_1	L_2	k_u	a_0	r_{1u}	r_{2u}	cos i_u	L_1	L_2
0.90	0.111	0.100	0.900	0.100	0.83	0.97	0.104	0.087	0.024	0.103	0.897
.80	.125	.100	.900	.100	.87	.84	.107	.094	.035	.119	.881
.70	.142	.100	.900	.100	.78	.73	.120	.094	.052	.137	.863
.60	.167	.100	.900	.100	.51	.83	.155	.078	.100	.120	.880
.50	.200	.100	.900	.100	.38	1.00	.186	.072	.114	.100	.900

unity. It is easy to show that this demands that a_0 in the first case, and $k_u^2 a_0$ in the second shall not be less than β. Actual computation of the values of k_u and a_0 consistent with the values of $\chi(k_u, a_0, \frac{1}{4})$, which best represent the form of the various light-curves, shows that the second condition can never be satisfied, while it is always possible to satisfy the first, provided that L_1 lies between unity and a certain limiting value determined by the fact that a_0 must not exceed unity. A summary of the results is as follows:

k_d	β	$\chi(k_u, a_0, \frac{1}{4})$	Limiting Values of		
			k_u	a_0	$1 - L_1$
1.00	1.000	2.085	0.844	1.000	0.000
0.90	0.917	2.037	0.861	0.917	0.000
			0.818	1.000	0.128
0.80	0.784	1.972	0.925	0.784	0.000
			0.776	1.000	0.246
0.70	0.636	1.850	0.880	0.636	0.000
			0.685	1.000	0.243
0.60	0.488	1.714	0.585	0.488	0.000
			0.554	1.000	0.184
0.50	0.350	1.588	0.000	0.37	0.000
			0.387	1.00	0.103

The principal interest of this table is the last column, which shows that, unless the "darkened" stars are of decidedly unequal brightness, it is impossible to get a satisfactory representation of the light-curve of the annular eclipse and the depth of the total eclipse on the assumption of uniformly illuminated disks. For example, if the stars are equal in radius ($k=1$) and eclipse one another centrally, the sum of the depths of the two minima will be equal to the whole light of the system. This condition is alone sufficient to prove that the stars are of equal radius; but on the hypothesis of uniform disks the value of $\chi(k_u, a_0, \frac{1}{4})$ must be 2.46, while for "darkened" stars it is 2.09—that is, the form of the light-curve will be quite different. Such systems, if carefully observed, may prove to be of great interest in the study of the actual existence and amount of darkening toward the limb.

The light-curves for a non-central eclipse of any pair of stars, "darkened" or "uniform," may be derived from that for central eclipse by diminishing all the values of $\sin^2\theta$ by a suitable constant. The considerations just developed for central annular eclipses therefore hold in general regarding the representation of light-curves, really due to eclipses of "darkened" stars, on the assumption of stars of uniform brightness. With a few exceptions (as noted above), a very close representation of the light-curve can be obtained, and very accurate observations would be necessary to distinguish between the two hypotheses.

4. *Determination of the elements in the case of annular and partial eclipses.*—As annular and partial eclipses of stars completely darkened at the limb cannot be distinguished by inspection of the light-curve, they must be discussed together. The same functional relation may be employed in the discussion of these eclipses as has been used in the previous papers. The quantity α, which represents the obscuration of light at any given phase, in terms of a standard value, might be defined here in more than one way. We will choose as unit the loss of light at the moment of internal tangency and, for clearness' sake, call the obscuration measured in these terms α''. α'' is therefore less than unity during the partial phases, and exceeds unity during the annular phases, reaching a maximum

for central eclipse, for which we may call its value $1+x$, x being a function of k. Having thus defined a'', we may proceed to determine the function $p(k, a'')$ which defines the distance of centers, and hence the functions $\psi(k, a_1'')$ and $\chi(k, a_0'', n)$ exactly as in the previous cases,[1] the only difference being that, in the tables, values of a'' greater than unity appear. For the p- and ψ-functions tabular values are given for intervals of $0.2x$ between $a_0''=1$ and $a_0''=1+x$. For the χ-functions this is not necessary, as computation shows that linear interpolation between the values for $a_0''=1$ and $a_0''=1+x$ gives the intermediate values with sufficient accuracy. As the empirical linear relations between the various χ-functions for different values of n, which existed in the case of uniform disks, no longer hold good here, three of these functions have been tabulated, for $n=0$, $n=\frac{1}{4}$, and $n=\frac{3}{4}$. The tabular intervals in a_0'', though very unequal, suffice for the drawing of a series of curves from which the values of the function may in all cases be taken.

Just as in the case of uniform disks, it is impracticable to derive the elements of the system from the light-curve of one of the minima alone, for this would make the determination of the critical data depend on quantities too small to be accurately found by observation. We must use the observed depths of both minima, and the form of the light-curve of the deeper one.

The relation connecting the depths of the minima with k and a_0 is now more complicated than in the case of uniform disks, for if the stars are of unequal radii the light-curves of the two minima (even if expressed in light-intensities rather than in magnitudes) are not similar in form; and one may show a constant phase when the other does not, even though the orbit is circular. If a_0 denotes the fraction of the light of the smaller star which is lost at the greatest phase of its eclipse behind the larger, we may call the fraction of the light of the larger star which is lost at the corresponding phase during the other eclipse, $a_0 Q(k, a_0)$—thus defining a new Q-function of k and a_0. This function may be immediately computed from the data of the previous tables, and it is given in Table V. For uniform disks it reduces to k^2. If, as previously, $1-\lambda_1$

[1] *Astrophysical Journal*, **35**, 321 and 326, 1912, or see below pp. 405, 406.

and $1-\lambda_2$ denote the losses of light at the two minima, and L_1, L_2 the light-emissions of the two stars, we have

$$\left.\begin{aligned} 1-\lambda_1 &= \alpha_0 L_2 \\ 1-\lambda_2 &= \alpha_0 L_1 Q(k, \alpha_0) \end{aligned}\right\} \qquad (1)$$

whence, (since $L_1+L_2=1$)

$$\alpha_0 = 1-\lambda_1+\frac{1-\lambda_2}{Q(k, \alpha_0)} \qquad (2)$$

or

$$Q(k, \alpha_0) = \frac{1-\lambda_2}{\alpha_0-(1-\lambda_1)} \qquad (3)$$

When $k=1$, $Q=1$, and $\alpha_0=1-\lambda_1+1-\lambda_2$ (as in the case of uniform disks)—which is the smallest value it can have consistently with the given data. For suitable values of α_0 between this limit and unity (e.g., the tabular values) we may compute $Q(k, \alpha_0)$ by (3)

TABLE Iy

VALUES OF $p(k, \alpha'')$ FOR ANNULAR ECLIPSES OF DISKS COMPLETELY DARKENED TOWARD THE EDGE

α''	$k=1.0$	$k=0.9$	$k=0.8$	$k=0.7$	$k=0.6$	$k=0.5$	$k=0.4$
0.00....	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000	+1.000
0.02....	0.796	0.786	0.778	0.772	0.768	0.765	0.762
0.05....	0.685	0.660	0.644	0.635	0.628	0.624	0.620
0.10....	0.543	0.511	0.495	0.485	0.479	0.473	0.468
0.15....	0.428	0.390	0.370	0.357	0.350	0.344	0.338
0.20....	0.328	0.287	0.263	0.250	0.240	0.232	0.226
0.25....	0.240	0.195	0.168	0.150	0.142	0.134	0.127
0.30....	0.158	0.109	+0.079	+0.060	+0.049	+0.042	+0.036
0.35....	0.080	+0.029	−0.003	−0.024	−0.036	−0.043	−0.048
0.40....	+0.004	−0.048	0.083	0.104	0.117	0.124	0.129
0.50....	−0.143	0.200	0.237	0.257	0.272	0.280	0.285
0.60....	0.284	0.341	0.379	0.401	0.413	0.421	0.427
0.70....	0.424	0.482	0.521	0.542	0.553	0.561	0.567
0.80....	0.570	0.627	0.663	0.682	0.693	0.700	0.706
0.85....	0.650	0.704	0.736	0.754	0.763	0.770	0.776
0.90....	0.737	0.787	0.814	0.828	0.835	0.842	0.847
0.95....	0.837	0.880	0.899	0.907	0.913	0.918	0.922
0.98....	0.910	0.943	0.955	0.959	0.962	0.964	0.967
1.00....	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000	−1.000
1+0.2x..	−1.000	−1.014	−1.030	−1.047	−1.070	−1.100	−1.145
1+0.4x..	1.000	1.030	1.062	1.099	1.147	1.216	1.312
1+0.6x..	1.000	1.048	1.100	1.163	1.241	1.355	1.516
1+0.8x..	1.000	1.069	1.146	1.246	1.361	1.533	1.780
1+x....	−1.000	−1.111	−1.250	−1.429	−1.667	−2.000	−2.500
x.....	0.000	0.015	0.047	0.084	0.143	0.220	0.322

TABLE IIy

VALUES OF $\psi(k, a_1'')$ FOR ANNULAR ECLIPSES OF DISKS COMPLETELY DARKENED TOWARD THE EDGE

a_1''	$k=1.0$	$k=0.9$	$k=0.8$	$k=0.7$	$k=0.6$	$k=0.5$	$k=0.4$
0.00....	+7.865	+7.920	+7.570	+6.961	+6.333	+5.648	+5.082
0.02....	6.115	6.161	5.900	5.445	4.979	4.472	4.027
0.05....	5.248	5.213	4.977	4.605	4.224	3.813	3.474
0.10....	4.215	4.181	4.021	3.747	3.472	3.152	2.876
0.15....	3.440	3.403	3.278	3.073	2.860	2.606	2.410
0.20....	2.825	2.790	2.692	2.530	2.360	2.160	2.003
0.25....	2.307	2.280	2.194	2.065	1.945	1.791	1.665
0.30....	1.858	1.835	1.772	1.668	1.574	1.455	1.369
0.35....	1.474	1.448	1.400	1.320	1.247	1.160	1.098
0.40....	1.115	1.099	1.061	1.007	0.953	0.891	0.849
0.45....	0.794	0.777	0.750	0.714	0.676	0.636	0.608
0.50....	0.504	0.484	0.470	0.454	0.432	0.404	0.388
0.55....	+0.240	+0.231	+0.227	+0.217	+0.209	+0.196	+0.188
0.60....	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.65....	−0.217	−0.211	−02.06	−0.200	−0.194	−0.187	−0.182
0.70....	0.411	0.406	0.397	0.388	0.376	0.367	0.358
0.75....	0.585	0.580	0.572	0.556	0.546	0.535	0.526
0.80....	0.740	0.735	0.723	0.715	0.706	0.697	0.689
0.85....	0.879	0.876	0.872	0.863	0.856	0.850	0.847
0.90....	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.95....	1.097	1.106	1.118	1.128	1.141	1.148	1.154
0.98....	1.138	1.158	1.181	1.202	1.223	1.232	1.234
1.00....	−1.156	−1.190	−1.224	−1.255	−1.284	−1.296	−1.308
$1+0.2x$.....		−1.196	−1.249	−1.309	−1.383	−1.460	−1.570
$1+0.4x$.....		1.201	1.273	1.362	1.483	1.630	1.845
$1+0.6x$.....		1.207	1.295	1.416	1.583	1.804	2.126
$1+0.8x$.....		1.211	1.316	1.470	1.683	1.976	2.414
$1+x$.....		−1.215	−1.335	−1.519	−1.790	−2.164	−2.746

and determine from Table V the value of k consistent with the observational data and each assumed value of a_0. We may then find from Table IIIx the values of the χ-functions corresponding to these pairs of values of k and a_0, and by comparison with the observed data (given by the equation $\chi(k, a_0, n)=\dfrac{\sin^2\theta(n)}{\sin^2\theta(\frac{1}{2})}$), we may determine what value of a_0 (with its corresponding value of k) gives the best representation of the observed curve, on the hypothesis that the principal minimum corresponds to the eclipse of the smaller star by the larger, whose depth is $1-\lambda_1$.

If, however, we assume that the principal minimum corresponds to the eclipse of the larger star by the smaller (of depth $1-\lambda_2$), we must express the maximum obscuration in terms of the unit

TABLE IIIx

VALUES OF THE χ-FUNCTIONS FOR TOTAL ECLIPSES OF DISKS COMPLETELY DARKENED TOWARD THE EDGE

$\chi(k, a_0, \frac{3}{4})$

k	$a_0=1.0$	$a_0=0.98$	$a_0=0.95$	$a_0=0.9$	$a_0=0.8$	$a_0=0.6$	$a_0=0.4$	$a_0=0.2$	$a_0=0.0$
1.00....	0.344	0.358	0.370	0.381	0.399	0.427	0.438	0.448	0.457
0.90....	0.410	0.406	0.408	0.410	0.418	0.434	0.442	0.450	0.457
0.80....	0.460	0.443	0.439	0.435	0.435	0.441	0.446	0.452	0.457
0.70....	0.499	0.470	0.461	0.455	0.450	0.448	0.450	0.454	0.457
0.60....	0.531	0.495	0.483	0.474	0.464	0.455	0.453	0.455	0.457
0.50....	0.560	0.522	0.504	0.492	0.476	0.460	0.456	0.457	0.457
0.40....	0.585	0.544	0.524	0.508	0.486	0.465	0.459	0.458	0.457
				$\chi(k, a_0, \frac{1}{4})$					
1.00....	2.086	2.065	2.026	1.977	1.922	1.854	1.814	1.768	1.707
0.90....	1.923	1.926	1.912	1.891	1.866	1.824	1.794	1.759	1.707
0.80....	1.789	1.818	1.822	1.819	1.814	1.800	1.778	1.750	1.707
0.70....	1.694	1.736	1.749	1.759	1.767	1.777	1.759	1.739	1.707
0.60....	1.615	1.665	1.689	1.705	1.732	1.753	1.740	1.727	1.707
0.50....	1.550	1.609	1.632	1.652	1.690	1.726	1.720	1.715	1.707
0.40....	1.498	1.558	1.578	1.610	1.653	1.691	1.699	1.703	1.707
				$\chi(k, a_0, 0)$					
1.00....	5.44	5.30	5.16	4.95	4.64	4.24	3.94	3.64	3.41
0.90....	4.64	4.61	4.58	4.48	4.32	4.06	3.84	3.60	3.41
0.80....	4.02	4.12	4.12	4.09	4.04	3.91	3.75	3.55	3.41
0.70....	3.58	3.72	3.76	3.77	3.80	3.75	3.63	3.50	3.41
0.60....	3.20	3.38	3.46	3.50	3.58	3.60	3.52	3.46	3.41
0.50....	2.90	3.10	3.19	3.26	3.37	3.46	3.44	3.42	3.41
0.40....	2.66	2.82	2.94	3.05	3.19	3.32	3.35	3.38	3.41

previously chosen for our tables—namely, the loss of light at internal tangency. At this moment in the other eclipse the smaller star is completely hidden, and $a_0=1$. Hence, by (1), $1-\lambda_2=L_1Q(k, 1)$—which is the unit to be used in measuring the obscuration a_0''. We have therefore

$$a_0''Q(k, 1)=a_0Q(k, a_0). \tag{4}$$

For each pair of values of k, a_0 previously found we may find a_0'' by (4), and then take the values of the χ-functions from Table IIIy, and find as before what system of values of k, a_0, a_0'' gives the best approximation to the observed values of these functions. Which of these cases corresponds to the principal minimum must be determined by trial. Usually only one of the two assumptions

TABLE III*y*

VALUES OF THE χ-FUNCTIONS FOR ANNULAR ECLIPSES OF DISKS COMPLETELY DARKENED TOWARD THE EDGE

$\chi(k, a_0'', \frac{3}{4})$

k	$a_0''=1+x$	$a_0''=1.0$	$a_0''=0.95$	$a_0''=0.9$	$a_0''=0.8$	$a_0''=0.6$	$a_0''=0.4$	$a_0''=0.2$	$a_0''=0.0$
1.00....	0.344	0.344	0.370	0.381	0.399	0.427	0.438	0.448	0.457
0.90....	0.361	0.365	0.378	0.386	0.401	0.422	0.434	0.446	0.457
0.80....	0.388	0.385	0.390	0.396	0.405	0.423	0.435	0.446	0.457
0.70....	0.430	0.409	0.407	0.411	0.415	0.425	0.437	0.447	0.457
0.60....	0.477	0.430	0.428	0.426	0.427	0.430	0.441	0.448	0.457
0.50....	0.526	0.448	0.445	0.442	0.440	0.435	0.444	0.450	0.457
0.40....	0.579	0.462	0.457	0.454	0.448	0.444	0.446	0.452	0.457

$\chi(k, a_0'', \frac{1}{4})$

1.00....	2.086	2.086	2.026	1.977	1.922	1.854	1.814	1.768	1.707
0.90....	2.086	2.073	2.026	1.983	1.925	1.855	1.814	1.768	1.707
0.80....	2.038	2.021	1.991	1.964	1.915	1.850	1.808	1.765	1.707
0.70....	1.923	1.941	1.936	1.924	1.883	1.840	1.800	1.760	1.707
0.60....	1.812	1.877	1.883	1.874	1.843	1.815	1.787	1.751	1.707
0.50....	1.685	1.814	1.820	1.814	1.801	1.788	1.770	1.743	1.707
0.40....	1.566	1.760	1.767	1.762	1.760	1.756	1.748	1.732	1.707

$\chi(k, a_0'', 0)$

1.00....	5.44	5.44	5.16	4.95	4.64	4.24	3.94	3.64	3.41
0.90....	5.50	5.44	5.22	5.02	4.71	4.31	4.03	3.70	3.41
0.80....	5.29	5.19	5.04	4.90	4.65	4.27	3.99	3.69	3.41
0.70....	4.77	4.82	4.74	4.64	4.46	4.17	3.92	3.66	3.41
0.60....	4.25	4.44	4.41	4.35	4.24	4.01	3.83	3.60	3.41
0.50....	3.68	4.08	4.09	4.06	4.00	3.88	3.74	3.55	3.41
0.40....	3.11	3.77	3.79	3.78	3.75	3.71	3.65	3.51	3.41

gives the computed values of the χ-functions in the neighborhood of the observed values.

The question whether a given eclipse is annular or partial can arise only when the larger of the two stars is considerably the brighter, and the smaller one is in front of it during the principal minimum, for otherwise the minimum at which total eclipse is possible will be deep enough to make it certain whether or not it shows a constant phase. When this minimum is so shallow that this question cannot be definitely decided, the procedure is similar to that just described for the case of a partial eclipse. If the principal eclipse is annular, the secondary must be total. We may therefore set $a_0=1$ in (3) and find the value of $Q(k, a_0)$ (remembering that $1-\lambda_1$ is now the loss of light at the secondary minimum).

The last line but one of Table V gives the values of this function for internal contact of the disks, and the last line its values for central eclipse, when the loss of light is $1+x$ times that at internal contact. In either of these cases the value of k corresponding to a given Q may be found, and the χ-functions taken out of Table III*y*, the other argument being $a_0''=1$ for grazing annular eclipse, and $a_0''=1+x$ for central annular eclipse.

TABLE V

VALUES OF $Q(k, a_0)$ FOR PARTIAL ECLIPSES OF DISKS COMPLETELY DARKENED TOWARD THE EDGE

a_0	$k=1.0$	$k=0.9$	$k=0.8$	$k=0.7$	$k=0.6$	$k=0.5$	$k=0.4$
0.00....	1.000	0.769	0.573	0.410	0.279	0.177	0.102
0.10....	1.000	0.777	0.585	0.426	0.292	0.188	0.108
0.20....	1.000	0.786	0.596	0.438	0.302	0.195	0.113
0.30....	1.000	0.796	0.605	0.446	0.310	0.200	0.116
0.40....	1.000	0.805	0.614	0.453	0.316	0.205	0.119
0.50....	1.000	0.814	0.624	0.461	0.321	0.210	0.123
0.60....	1.000	0.823	0.634	0.470	0.328	0.215	0.126
0.70....	1.000	0.832	0.646	0.483	0.339	0.222	0.131
0.80....	1.000	0.844	0.661	0.499	0.352	0.232	0.137
0.90....	1.000	0.861	0.685	0.522	0.371	0.247	0.147
0.95....	1.000	0.872	0.704	0.538	0.384	0.256	0.153
0.98....	1.000	0.880	0.717	0.553	0.398	0.267	0.160
0.99....	1.000	0.890	0.728	0.563	0.405	0.273	0.164
1.00....	1.000	0.904	0.750	0.587	0.427	0.289	0.175
$1+x$....	1.000	0.917	0.784	0.636	0.488	0.351	0.230

For eclipses whose maximum phase lies between these limits, with $a_0''=1+mx$ (where m is any fraction), the values of the Q- and χ-functions can be found by linear interpolation between those given in the tables for 1 and $1+x$, and the same procedure then followed. Comparison of the χ-functions so computed with those derived from the observed curve will show whether the curve can be explained by an annular eclipse of darkened stars, and, if so, whether this eclipse is grazing, central, or intermediate between these limits. When for the eclipse so computed k exceeds 0.8 (approximately), the solution is unique, i.e., there exists no other pair of values of k, a_0 which will lead to the same depths of minima and the same values of χ for the principal minimum; but for smaller values of k there are two solutions, one giving an annular,

or large partial, eclipse, the other giving a smaller partial eclipse with stars more nearly equal. In some cases the difference in the computed curves for secondary minimum could distinguish which of these solutions is correct; in others the ambiguity must remain.

The condition that two solutions may be possible may also be put in the form $1-\lambda_2<0.78\lambda_1$—that is, the loss of light at principal minimum must be less than 78 per cent of the light remaining at the secondary. When both eclipses are shallow (say less than $0^m.4$), and nearly equal in depth, the solution becomes almost indeterminate, mainly because the observed values of χ cannot be read off with any precision from the light-curve. If the relative brightness of the components is known from spectrographic measures, the solution remains definite for even the smallest eclipses.

When the best values of k and a_0 or a_0'' have been determined, it is easy to fix upon such a value of $\sin^2\theta(\frac{1}{2})$ that the computed values of $\sin^2\theta$ for the beginning and end of eclipse, and for the moments when the loss of light is $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the maximum, agree very closely with those derived from the observed curve. Unless the latter is very precisely known, it is unnecessary to go farther, and these computed points may be joined by a free-hand curve. If greater accuracy, however, is desired, the values of L_1 and L_2 may be computed by (1), and then values of $\sin^2\theta$ corresponding to a loss of light of a_1L_2 for the eclipse of the smaller star by the larger, or of $a_1''\,L_1Q(k, 1)$ for the other eclipse be determined from the observed light-curve in the usual fashion. We then have for the values 0.0, 0.1, etc., of a_1 (or a_1''), as in former cases, $\sin^2\theta=A+B\psi(k, a_1)$, where A and B are constants, and the function ψ is taken from Table II*x* if the larger star is in front, or from Table II*y* if it is behind the other. The fact that $a_1=a_0$ when $\theta=0$ gives the relation $A+B\psi(k, a_0)=0$. It is easy to determine A and B—graphically or otherwise—so that the series of equations thus formed and in particular the last one are as accurately represented as possible (with consideration of their varying weights), and a precise light-curve may thus be determined.

When once the value of θ for the beginning or end of eclipse,

which we may call θ', has been computed, the remaining elements may be found at once from the equations

$$\left.\begin{aligned} r_1^2(1+k)^2 &= \cos^2 i \cos^2\theta' + \sin^2\theta' \\ \cos^2 i &= r_1^2\{1+kp(k, \alpha_0)\}\,, \end{aligned}\right\} \quad (5)^*$$

* *Astrophysical Journal* **35**, 331.

where $p(k, \alpha_0)$ is to be taken always from Table Ix, except in the case of annular eclipse, when we must substitute $p(k, \alpha_0'')$ from Table Iy. When the light-curve of the principal minimum has been computed, that of the secondary may be found, if the orbit is circular, by computing the χ-functions for this eclipse. As the depth and duration of this minimum are already known, three points on each branch of the light-curve may now be computed—which will almost always be sufficient. This method fails if the principal eclipse is annular, and the secondary total, in which case the constants A and B of the light-curve may be found with the aid of Table IIax, and then the light-curve itself by Table IIx, as described above.

The problem of determining the elements of the system when the stars are completely darkened toward the limb is now solved for a circular orbit. For the modifications necessary when the orbit is elliptical reference may be made to a preceding paper (*Astrophysical Journal*, **36**, 54–60) especially to the method described on p. 58, bearing in mind that the equation

$$\frac{1-\lambda_1}{\alpha_1}+\frac{1-\lambda_2}{\alpha_2 Q(k, \alpha_2)}=1$$

must be substituted for equation (33) of that discussion. The determination of $e \sin\omega$ from the light-curve is always a difficult one and incapable of great precision.

When the elements of the elliptical orbit are known from spectroscopic observations (as is now or soon will be the case for the brighter eclipsing variables), it is best to go back to first principles, using the equation for the distance of centers at any time

$$r^2\{\cos^2 i+\sin^2 i \cos^2(v+\omega)\}=r_1^2\{1+kp(k, \alpha)\}^2$$

where r denotes as usual the radius vector and v the true anomaly. With the aid of the spectroscopic data the first member of this

equation at any instant, may be expressed in the form $a+b\cos^2 i$, where a and b are known. If k and a_0 are known, the second member becomes a known multiple of r_1^2, and the resulting equations may be solved for r_1^2 and $\cos^2 i$. To find k and a_0, when one of the minima is sufficiently deep to permit the determination of reliable values of χ from the light-curve, we may follow the method cited in the last paragraph, which is much simplified when $e \sin \omega$ is known. Sometimes the spectrographic data themselves make it possible to estimate the relative brightness of the two stars—that is, to find L_1 and L_2. We may then find the values of a corresponding to any observed magnitude at the two eclipses (remembering that during the eclipse when the small star of light L_2 partially obscures the other the loss of light corresponding to $a=1$ is k^2L_1 for uniform disks and $L_1Q(k, 1)$ for "darkened" disks), and the problem is thus further simplified and may be satisfactorily solved, even when the eclipses are too shallow to allow a good observational determination of χ.

5. *Ellipticity of the stars.*—The light variations due to the changing presentation of a rotating ellipsoid, whose apparent surface brightness varies as the cosine of the angle of emission of its light, may readily be calculated. If $d\sigma$ denotes an element of area of the surface, and γ the angle between the visual ray and the normal, the apparent projected area of the element is $d\sigma \cos \gamma$, and its apparent surface brightness $J_0 \cos \gamma$, where J_0 is a constant. The whole amount of the light received by the eye is therefore

$$L = J_0 \int \cos^2 \gamma d\sigma , \qquad (6)$$

where the integral is to be taken over the visible half of the surface. Since the ellipsoid is symmetrical, and the integrand is everywhere positive, this integral is one-half the integral of $\cos^2 \gamma d\sigma$ over the whole surface—which much simplifies the problem by removing annoying boundary conditions.

Let us now take the principal axes of the ellipsoid as our co-ordinate axes, and let l, m, n be the direction cosines of the normal to the element $d\sigma$, and λ, μ, ν those of the observer's direction. Then $\cos \gamma = l\lambda + m\mu + n\nu$. Introducing this into (6), the integral splits up at once into six parts, of which the three of the

form $2\lambda\mu\int lmd\sigma$ vanish on account of the symmetry of the ellipsoid about the co-ordinate planes, so that we have

$$L=\tfrac{1}{2}J_0(\lambda^2\int l^2d\sigma+\mu^2\int m^2d\sigma+\nu^2\int n^2d\sigma)\,,$$

where the integrals are to be taken over the whole surface of the ellipsoid. If i is the inclination of the line of sight to the equatorial plane of the ellipsoid (taken here as the xy-plane), and u the position angle of the longest axis of the ellipsoid in this plane at any time, we have

$$\lambda=\sin i\cos u\,,\quad \mu=\sin i\sin u\,,\quad \nu=\cos i\,,$$

and hence

$$L=A\sin^2 i\cos^2 u+B\sin^2 i\sin^2 u+C\cos^2 i \qquad (7)$$

where $A=\tfrac{1}{2}J_0\int l^2d\sigma$, ... etc. As the ellipsoid rotates, u increases uniformly with the time. The light-curve is therefore a simple sine-curve of amplitude proportional to $\sin^2 i$. If L_0 denotes the maximum light (which we may suppose to occur when $u=90^\circ$), we have $L_0-L=(A-B)\sin^2 i\cos^2 u$. In the case of uniform disks we derived an equation which was identical in form except that its first member was $L_0^2-L^2$.

We have now to determine the constants A, B, C. Let the semiaxes of the ellipsoid be a, b, c. The co-ordinates of any point on the surface may be represented by the equations

$$x=a\cos\theta\,,\qquad y=b\sin\theta\cos\phi\,,\qquad z=c\sin\theta\sin\phi\,,$$

where θ and ϕ are auxiliary angles. The direction cosines of the normal to the surface at this point are proportional to $\frac{x}{a^2}$, $\frac{y}{b^2}$, $\frac{z}{c^2}$: whence we find

$$l^2=\frac{\dfrac{\cos^2\theta}{a^2}}{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta\cos^2\phi}{b^2}+\dfrac{\sin^2\theta\sin^2\phi}{c^2}}\,,\ \text{etc.}$$

If we set $a^2=b^2(1+\eta_1)$, $a^2=c^2(1+\eta_2)$ we may write

$$l=\frac{\cos\theta}{\sqrt{1+\eta_1\sin^2\theta\cos^2\phi+\eta_2\sin^2\theta\sin^2\phi}}\,.$$

$ld\sigma$, which is the projection of the element of area on the plane $x=0$, is easily found to be $bc \sin\theta \cos\theta d\theta d\phi$. We have, therefore, for the constant A

$$A=\tfrac{1}{2}J_0 bc\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \frac{\cos^2\theta \sin\theta}{\sqrt{1+\eta_1 \sin^2\theta \cos^2\phi+\eta_2 \sin^2\theta \sin^2\phi}}.$$

In practice η_1 and η_2 are fairly small. We may therefore expand the radical as powers of these quantities. Integrating term by term, we find easily

$$A=\tfrac{2}{3}\pi bcJ_0\{1-\tfrac{1}{10}(\eta_1+\eta_2)+\tfrac{3}{280}(3\eta_1^2+2\eta_1\eta_2+3\eta_2^2)+ \ldots .\}$$

when, as above,

$$\eta_1=\frac{a^2-b^2}{b^2}, \qquad \eta_2=\frac{a^2-c^2}{c^2}.$$

If we wish to express the other constants in terms of the quantities η_1, η_2, we have

$$\frac{b^2-a^2}{a^2}=\frac{-\eta_1}{1+\eta_1}=-\eta_1+\eta_1^2+ \ldots ,$$

$$\frac{b^2-c^2}{c^2}=\frac{\eta_2-\eta_1}{1+\eta_1}=\eta_2-\eta_1+\eta_1^2-\eta_1\eta_2 \ldots ,$$

and hence

$$B=\tfrac{2}{3}\pi acJ_0\{1-\tfrac{1}{10}(\eta_2-2\eta_1)-\tfrac{1}{280}(32\eta_1^2-4\eta_1\eta_2-9\eta_2^2) \ldots \}.$$

Setting $b=a(1+\frac{1}{2}\eta_1-\frac{1}{8}\eta_1^2 \ldots)$, we find

$$B=A(1+\tfrac{4}{5}\eta_1-\tfrac{16}{175}\eta_1^2+\tfrac{4}{175}\eta_1\eta_2 \ldots).$$

The expression for C may be found by interchanging η_1 and η_2 in this equation.

For a prolate spheroid of revolution, if b is the longest axis and ϵ the eccentricity of a meridian section, we have $\eta_1=-\epsilon^2$, $\eta_2=0$, and hence

$$B=A(1-\tfrac{4}{5}\epsilon^2-\tfrac{16}{175}\epsilon^4 \ldots).$$

For uniform apparent brightness we had $B=A\sqrt{1-\epsilon^2}=A(1-\frac{1}{2}\epsilon^2-\frac{1}{8}\epsilon^4 \ldots)$. The amplitude of the light variation measured as a function of the maximum light is therefore increased by darkening in the ratio of $\frac{8}{5}-\frac{19}{175}\epsilon^2 \ldots$, so that neglect of this influence, if it exists, will lead to decidedly too great a value for the ellipticity of the stars.

The exact calculation of the light-curve arising from the mutual eclipse of a pair of such darkened ellipsoids would be very complicated, for the contours of equal percentages of darkening on the apparent disk are not in general elliptical, nor is the brightest point of the disk coincident with its apparent center (unless the ellipsoid is viewed from the direction of one of its axes). Even in this case, the apparent brightness of the disk of a prolate spheroid, seen endwise, will fall off more rapidly toward the limb than in the case of a sphere for which the relation between the apparent brightness and the inclination of the surface to the line of sight is the same.

Neglecting all these things, we may, as an approximation, suppose that the distribution of apparent brightness on the elliptical disk may be obtained simply by "stretching" to a suitable degree, that already studied on the disk of a sphere, and transforming the circular contours of equal brightness into similar ellipses. As the apparent disks during actual eclipses are nearly circular, this will probably be a fair approximation. With this limitation we may proceed to a solution as follows: The equation of the non-eclipse portion of the intensity-curve should be of the form $L=L_0(1-Z\ \cos^2\ \theta)$, where $Z=(\frac{1}{5}\epsilon^2+\frac{1}{1}\frac{6}{7}\frac{}{5}\epsilon^4\ \ldots\ )\ \sin^2\ i$. By plotting L against $\cos^2\ \theta$, the value of Z may be found, and by subtracting the change in stellar magnitude due to ellipticity from the observed magnitudes, a "rectified" curve may be obtained just as in the case where the stellar disks are assumed to be of uniform brightness.

The rectified curve may be discussed as already described, using the tables for disks darkened at the limb, but the equations appropriate for elliptical disks—

$$\left.\begin{aligned}\psi(k,\ a_1)&=\frac{(\sin^2\theta_1-\sin^2\theta_2)(1-z\cos^2\theta_3)}{(\sin^2\theta_2-\sin^2\theta_3)(1-z\cos^2\theta_1)},\\ \chi(k,\ a_0,\ n)&=\frac{\sin^2\theta(n)(1-z\cos^2\theta(\tfrac{1}{2}))}{\sin^2\theta(\tfrac{1}{2})(1-z\cos^2\theta(n)}.\end{aligned}\right\}*$$

* *Astrophysical Journal*, **36**, 65–66, 1912.

The constant z appearing in these equations is $\epsilon^2\sin^2 i$. From the above value of Z we find with sufficient approximation $z=\frac{5}{4}Z-\frac{5}{2}\frac{}{8}Z^2$.

6. *Intermediate degrees of darkening.*—To discriminate between different degrees of darkening at the limb, we must have a precise determination of the whole light-curve (including the non-eclipse portions as well as the secondary minimum) of either (*a*) a star with a conspicuous constant phase at principal minimum (the shallower the better), or (*b*) a star in which the sum of the losses of light at the two minima (after correction for ellipticity of the stars and for the effects of their mutual radiation) is nearly equal to the whole light of the system. Many eclipsing variables satisfy the first requirement, and *RT Lacertae* appears to be an example of the second.[1] But unfortunately not one of these has yet been subjected to a sufficiently extensive study to define its whole light-curve with adequate precision. There is, therefore, no urgent reason for the computation of tables, etc., for intermediate degrees of darkening. Their construction would be a mere matter of reckoning, since the values of a, considered as a function of k and p, for any intermediate degree of darkening, may be found by linear interpolation between those for zero and complete darkening, which have already been computed. The values of p as a function of k and a, and hence those of the other functions involved in our discussions, may then be computed for any desired degree of darkening.

To do this for several degrees of darkening would involve much work, and an indirect method of solution appears to promise less labor. For uniform disks the loss of light during the progress of eclipse varies proportionally to the eclipsed area (measured as usual in terms of the area of the smaller disk). For disks of the same total light-emission, but darkened toward the limb, the loss of light is no longer proportional to the eclipsed area, but may be derived from it by multiplication by a certain factor, which depends on the relative radii of the stars, the eclipsed area, and the law of darkening. For the general law of darkening already assumed, $J=J_0(1-x+x\cos i)$, in which x measures the degree of darkening, this factor will evidently be a linear function of x, which becomes unity when $x=0$. We may therefore write,

$$\left.\begin{aligned} 1-l_1&=aL_2\{1+xX(k,a)\}\,,\\ 1-l_2&=k^2aL_1\{1+xY(k,a)\}\,,\end{aligned}\right\} \qquad (8)$$

[1] See the light-curve by Luizet, *Bulletin Astronomique*, **27**, 309, 1910.

where the notation is our usual one, except that a here denotes the eclipsed *area* in terms of the area of the smaller star, and X and Y are two new functions. The first equation holds good when the large star eclipses the smaller, the other in the opposite case. The functions X and Y may easily be computed with the aid of Tables I, Ix, and Iy. If a, a', and a'' correspond to identical values of k and p in these three tables, then

$$1+X(k, a)=\frac{a'}{a}, \qquad 1+Y(k, a)=\frac{a''Q(k, 1)}{ak^2}.$$

If now k and L_1 (or L_2) are known, the equations (8) can be used to compute the values of l, that is, of the light-intensity, corresponding to any desired eclipsed area a, upon any desired assumption as to the degree of darkening. The values of $\sin^2\theta$ corresponding to given values of a may then be found as usual from the light-curve and may be discussed by the usual method, using the tables computed for uniform disks. The value of k so obtained must of course agree with that originally assumed if the latter is correct.

An approximate value of k for any intermediate degree of darkening may be obtained by simple interpolation between the values obtained by solutions for the cases of zero and complete darkening made as already described. When the passage from one of these solutions to the other inverts the relative sizes of the two stars, this interpolation should be performed for r_1 and r_2 separately and their ratio then taken. When one of the eclipses is total (i.e., has a constant phase) the values of L_1 and L_2 may be obtained at once. Otherwise their values, together with that of a_0, the maximum eclipsed area, may be obtained from the observed depths $1-\lambda_1$ and $1-\lambda_2$ of the two minima, with the aid of the assumed value of k. Assuming different values of a_0 the equations

$$1-\lambda_1=a_0L_2\{1+xX(k, a_0)\}$$
$$1-\lambda_2=k^2a_0L_1\{1+xY(k, a_0)\}$$

may be solved for L_1 and L_2, and the correct value of a_0 is that which makes $L_1+L_2=1$.

If the value of k found after a solution on the lines just discussed does not agree with the value originally assumed, a second solution

may be made with a different assumed value of k, and exact agreement of the initial and final values of k obtained by the method of "false position."

This mode of solution is applicable to any desired degree of darkening. Though less elegant theoretically than a direct solution based on the construction of separate tables for each degree

TABLE VI

VALUES OF $X(k, a)$

a	k=1.0	k=0.9	k=0.8	k=0.7	k=0.6	k=0.5	k=0.4
0.10	−0.256	−0.261	−0.266	−0.270	−0.276	−0.282	−0.286
0.20	0.117	0.125	0.131	0.137	0.142	0.146	0.151
0.30	−0.020	−0.033	−0.042	−0.049	−0.056	0.062	0.068
0.40	+0.035	+0.022	+0.013	+0.005	−0.002	−0.008	−0.015
0.50	0.072	0.060	0.052	0.044	+0.038	+0.032	+0.027
0.60	0.097	0.083	0.073	0.067	0.062	0.058	0.055
0.70	0.103	0.090	0.082	0.074	0.069	0.066	0.064
0.80	0.088	0.079	0.073	0.067	0.064	0.062	0.060
0.90	0.058	0.051	0.047	0.045	0.044	0.043	0.042
0.95	0.035	0.030	0.028	0.027	0.026	0.026	0.025
0.98	+0.016	+0.015	+0.014	+0.014	+0.014	+0.013	+0.013
1.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000

VALUES OF $Y(k, a)$

0.10	−0.256	−0.297	−0.341	−0.386	−0.435	−0.490	−0.552
0.20	0.117	0.154	0.191	0.228	0.277	0.332	0.399
0.30	−0.020	−0.051	0.093	0.138	0.190	0.251	0.322
0.40	+0.035	+0.019	−0.020	0.068	0.126	0.194	0.268
0.50	0.072	0.067	+0.033	−0.011	0.071	0.139	0.209
0.60	0.097	0.105	0.075	+0.031	−0.024	0.086	0.160
0.70	0.103	0.127	0.108	0.075	+0.025	−0.036	0.111
0.80	0.088	0.137	0.134	0.111	0.066	+0.010	0.064
0.90	0.058	0.130	0.150	0.140	0.107	0.057	−0.010
0.95	0.035	0.122	0.156	0.160	0.133	0.087	+0.022
0.98	+0.016	0.117	0.161	0.176	0.156	0.114	0.050
1.00	0.000	+0.116	+0.172	+0.198	+0.186	+0.152	+0.093
Central annular	0.000	+0.145	+0.225	+0.298	+0.356	+0.404	+0.437

of darkening, it will probably prove in practice to involve much less work than the construction of such tables, unless a large number of stars had to be discussed—a contingency which at present seems remote, since there is not now a single star whose light-curve is known accurately enough to distinguish between different degrees of darkening with certainty. The values of the functions $X(k, a)$ and $Y(k, a)$ have been computed, and may be found in Table VI.

SUMMARY OF THE THEORY OF ECLIPSING VARIABLES

The present paper completes a series which discusses in a fairly exhaustive manner the determination of the elements of eclipsing variables, and which, it is hoped, may be of use for many years to come in the practical solution of this problem. As the formulae and tables are scattered through four numbers of this *Journal*, it may be appropriate to give here a brief summary of the notation and methods used together with directions for the computer. References throughout are to volume and page of this *Journal*.

I. GENERAL NOTATION

The unit of length is the radius of the relative orbit; the unit of light is the combined light of the two stars; the unit of mass is their combined mass. The subscript 1 refers always to the larger star. The principal quantities used are defined as follows:

Spherical Stars

r_1, radius of large star
$r_2 = kr_1$, radius of small star
δ, apparent distance of centers
i, inclination of the orbit
t, time from principal minimum in days
P, period of revolution in days
$\theta = \frac{2\pi t}{P}$, true longitude in orbit
$\frac{J_1}{J_2}$, ratio of surface brightness
ρ_1, ρ_2, densities of the stars
l, light at any time
L_1, light of large star
$L_2 = 1 - L_1$, light of small star
λ_1, minimum light at eclipse where large star is in front
λ_2, minimum light, small star in front
α, loss of light at any time in terms of loss for complete eclipse
α_0, greatest loss at minimum
n, fraction of greatest loss
y, mass of larger star

Elliptical Stars

a_1, a_2, semi-major axes
b_1, b_2, semi-minor axes
ϵ, eccentricity of meridian section
$z = \epsilon^2 \sin^2 i$

Darkened Stars

x, coefficient of darkening, **36**, 240
Z, defined by $z = \frac{5}{4} Z - \frac{5}{24} Z^2$
α'', loss of light at any time during an eclipse where the small star is in front, in terms of the loss at the moment of internal tangency.

II. FUNCTIONAL NOTATION AND INDEX OF TABLES

Function	Use	Where Found		
		Uniform	Darkened	
			Big Star in Front	Small Star in Front
$p(k, a)$, **35**, 332	Relation between loss of light and distance of centers	Table I, **35**, 333	Table Ix, **36**, 243	Table Iy, **36**, 390
$\Psi(k, a_1)$, **35**, 321	To find light-curve when L_1 and L_2 are known	Table II, **35**, 335	Table IIx, **36**, 245	Table IIy **36**, 391
$\Phi(k, a)$, **36**, 254	To find light-curve for central total eclipse	Table IV, **36**, 252	Table IVx, **36**, 253	
$\phi_1(k)$, $\phi_2(k)$, **35**, 336	To find the elements when L_1 and L_2 are known	Table IIa, **35**, 337	Table IIax, **36**, 246	
$\chi(k, a_0, \frac{1}{4})$, etc. **35**, 326	To find elements in case of partial eclipse	Table III, **35**, 337	Table IIIx, **36**, 392	Table IIIy, **36**, 393
$\omega_1(n)$, $\omega_2(n)$, **35**, 329	To find light-curve in case of partial eclipse (highly approximate)	Table IIIa, **35**, 338		
$Q(k, a_0)$, **36**, 389	To aid in solution of partial eclipse of completely darkened stars		Table V **36**, 394	
$X(k, a)$, $Y(k, a)$ **36**, 402	To aid in discussion of intermediate degrees of darkening		Table VI **36**, 403	

Table A is for converting stellar magnitudes into light intensities: **35**, 339.

Table B is for converting θ into $\sin\theta$: **35**, 339.

Table C is for converting elements derived on the "uniform" hypothesis into "darkened" elements (highly approximate): **36**, 250.

III. FUNDAMENTAL EQUATIONS FOR SPHERICAL STARS

The references to the original discussion precede the equations. θ_1 corresponds to $a=a_1$, θ_2 to $a=0.6$, θ_3 to $a=0.9$, $\theta(n)$ to $a=na_0$, θ' to $a=0$, and θ'' to $a=1$.

35, 321 $\delta^2=\cos^2 i+\sin^2 i\sin^2\theta$ ..... (*a*)

35, 321 $\delta=r_1\{1+kp(k, a)\}$ ..... (*b*)

35, 321 $A=\sin^2\theta_2$, $B=\sin^2\theta_2-\sin^2\theta_3$, $\sin^2\theta_1=A+B\psi(k, a_1)$ .... (*c*)

35, 323 $\cot^2 i=\dfrac{B}{\phi_2(k)}-A$, $r_1^2\operatorname{cosec}^2 i=\dfrac{B}{\phi_1(k)}$ ..... (*d*)

36, 254 $\sin^2\theta_1 = A\Phi(k, a_1)$ ..... (*e*)

35, 326 $\dfrac{\sin^2\theta(n)}{\sin^2\theta(\frac{1}{2})} = \chi(k, a_0, n)$ ..... (*f*)

35, 329 $\sin^2\theta(n) = C\omega_2(n) + D\omega_1(n)$, $\chi(k, a_0, \frac{1}{4}) = \dfrac{C}{D}$ ..... (*g*)

35, 331 $\left.\begin{aligned}\cos^2 i\cos^2\theta' + \sin^2\theta' &= r_1^2(1+k)^2\\ \cos^2 i &= r_1^2\{1+k\dot{p}(k, a_0)\}^2\end{aligned}\right\}$ ..... (*h*)

For Uniform Disks	For Darkened Disks
$a_0 = 1-\lambda_1+\dfrac{1-\lambda_2}{k^2}$ ..... (*j*)*	$Q(k, a_0) = \dfrac{(1-\lambda_2)}{a_0-(1-\lambda_1)}$ ..... (*n*)
$l_1 = 1-aL_2$, $1-\lambda_1 = a_0L_2$ ..... (*k*)†	$l_1 = 1-aL_2$, $1-\lambda_1 = a_0L_2$; a_0 from Table Ix ..... (*p*)
$l_2 = 1-k^2(1-l_1)\dfrac{L_1}{L_2}$ ..... (*m*)	$l_2 = 1-Q(k, a)\dfrac{L_1}{L_2}(1-l_1) =$ $1-a_0''L_1Q(k, 1)$ ..... (*q*)
	$a_0''Q(k, 1) = a_0Q(k, a_0)$ ..... (*r*)

$$\rho_1 = 0.01344\frac{y}{P^2r_1^3}, \qquad \rho_2 = 0.01344\frac{1-y}{P^2r_2^3} \quad \text{..... (s)‡}$$

$$\frac{J_1}{J_2} = \frac{k^2L_1}{L_2} \quad \text{..... (t)}$$

* *Astrophysical Journal*, **35**, 320.
† *Ibid.*, 318.
‡ *Ibid.*, **36**, 73.

IV. DIRECTIONS FOR THE COMPUTER

Preliminary: Reduce the observed curve to intensity-curve, taking maximum intensity as unity. Determine the epoch of minimum, and draw the best symmetrical curve to represent it.

A) The light is constant between minima, therefore the stars are spherical.

1) There is a constant phase at principal minimum of brightness λ_1. Then $L_1 = \lambda_1$, and $L_2 = 1-\lambda_1$.

Solution on uniform hypothesis: For tabular values of a_1 compute l_1 by (*k*), read θ from the light-curve, take sin θ from Table B, find $\psi(k, a_1)$ from (*c*), hence k from Table II, and other elements from (*d*) and Table II*a*. If $\cot^2 i$ comes out negative, use (*e*) and Table IV. Compute the light-curve for principal minimum from (*c*), (*k*), and Table II, and for the secondary minimum from (*m*).

For completely darkened stars substitute Tables II*x*, II*ax*, and IV*x*, for II, II*a*, and IV; and compute secondary minimum by (*q*) and Table V.

2) There is no constant phase at principal minimum, therefore the eclipses are partial (or perhaps the principal eclipse is annular, if the stars are darkened). Knowledge of the secondary minimum is essential in this case.

For uniform stars use (k) and the light-curve to find $\theta(n)$, then find C and D from (g), using Table III*a*. For assumed values of a_0 compute k from (j) and $\chi(k, a_0, \frac{1}{4})$ from Table III, and find what value of a_0 gives $\chi=\frac{C}{D}$. Then compute θ' from (g) (setting $n=0$), the elements from (h), using Table I, and L_2 from (k).

For darkened stars find $\theta(n)$ as above, then $\chi(k, a_0, \frac{3}{4})$, $\chi(k, a_0, \frac{1}{4})$, and $\chi(k, a_0, 0)$ from (f). For assumed values of a_0 find k from (n) and Table V, and a_0'' from (r). Then take the χ-functions from Table III*x*, with arguments k, a_0, if the principal minimum is assumed to correspond to λ_1, or from Table III*y*, with arguments k, a_0'', if to λ_2, and find what values of a_0 and k give the best agreement. Compute the other elements by (h), using Table I*x*, and find L_2 from (p). For details regarding annular eclipses see **36**, 393 ff.

B) The light is not constant between minima, therefore the stars are elliptical. Rectify the intensity-curve: (1) for ellipticity of the stars, **36**, 64, 65 (illustration, **36**, 139 ff.); (2) for radiation-effect (if necessary), **36**, 67–69. Then discuss as under *A*) except that for equations (c), (d), (f), (g), (h) must be substituted, respectively,

$$\psi(k, a_1)=\frac{(\sin^2\theta_1-\sin^2\theta_2)(1-z\cos^2\theta_3)}{(\sin^2\theta_2-\sin^2\theta_3)(1-z\cos^2\theta_1)} \quad \ldots\ldots (c')^*$$

$$\left.\begin{aligned}\cos^2 i+\sin^2 i\sin^2\theta' &= a_1^2(1-z\cos^2\theta')(1+k)^2\\ \cos^2 i+\sin^2 i\sin^2\theta'' &= a_1^2(1-z\cos^2\theta'')(1-k)^2\\ \cos^2 i &= a_1^2(1-z)\{1+kp(k, a_0)\}^2\end{aligned}\right\} \quad \ldots\ldots (d' \text{ and } h')\dagger$$

$$\frac{\sin^2\theta(n)\{1-z\cos^2\theta(\frac{1}{2})\}}{\sin^2\theta(\frac{1}{2})\{1-z\cos^2\theta(n)\}}=\chi(k, a_0, n) \quad \ldots\ldots (f')\ddagger$$

$$\frac{\sin^2\theta(n)}{1-z\cos^2\theta(n)}=C\omega_2(n)+D\omega_1(n)\,, \quad \frac{C}{D}=\chi(k, a_0, \tfrac{1}{4}) \quad \ldots\ldots (g')$$

* *Astrophysical Journal*, **36**, 65.

† *Ibid.*, 66.

‡ *Ibid.*, 66.

C) For the discussion of eccentric orbits refer to **36**, 54–60, 396–397, and the illustration, **36**, 146 ff. For the discussion of intermediate degrees of darkening refer to pp. 401–403, and Table VI of this paper.

PRINCETON UNIVERSITY OBSERVATORY
October 26, 1912

Reviews

Bijdrage tot de Kennis van Lijnenspectra. By T. Van Lohuizen. (Doctor's dissertation.) University of Amsterdam. The Hague. Printed by the "Transvalia" press, 1912. Pp. 102.

The author, in this thesis, (1) examines existing spectrum-series formulae; (2) tries adding more terms to the formula of Kayser and Runge; (3) devises a "new formula"; (4) tests this by series in *Li*, *Na*, and *Al;* (5) seeks to unite, in one connected system, all the series of an element; and (6) proposes several new series in the spectra of tin and antimony.

In the first section, he concludes that none of the formulae mentioned (Kayser and Runge, Rydberg, Ramage, Fowler and Shaw, Ritz, Mogendorff and Halm) are entirely satisfactory. In the second, he finds that the formula of Kayser and Runge is still unsatisfactory, even when another term, involving a fourth adjustable constant, is added. He accordingly starts with the formula of Rydberg, and seeks, in the third section, to find a modification of this which will be applicable to series in general. Curves are drawn on a large scale for different series, the wave-numbers of the lines being chosen as ordinates, and the successive integers as abscissae. A study of such curves leads him to the conclusion (fully developed in section 5) that the shape of the curve is the same for all series, even for the Balmer series of Hydrogen, but that the curve is shifted to different axes for different series, often with a slight rotation. He expresses this analytically by starting with the Balmer formula, and changing the axes by a translation along each one, and a rotation around the origin. The new expression so obtained involves a continued fraction, and this is reduced and simplified by assuming the rotation to be small, eventually taking the form:

$$y=b+(x-a)\gamma-\cfrac{N}{\left\{x-a-\frac{\gamma N}{x^2}\left(1-\frac{3a}{x}\right)\left(1+\frac{a}{x}\right)\right\}^2}$$

y is the wave-number of the series line; x is the integer, 1, 2, 3, etc.; a and b are the two translations, and γ is the angle of rotation; N is the universal series-constant.

The author does not call attention to the fact that this formula is not convergent; and the convergency of spectrum series is one of the strongest articles of faith of the spectroscopist, in spite of the fact that we cannot follow any series out to its limit. In the above geometrical scheme, *any* rotation makes the series ultimately divergent, and must therefore be thrown out of consideration.

It is true that the author does not use the formula given above, but drops the divergent part of it (the second term), the reason being given that only a few lines of any series are ever observed, and that the rotation is usually very small, especially so (as it happens) for cases like the principal series of *Na*, for which the most terms are known. With this further simplification, the formula runs:

$$y=b-\frac{N}{(x+a+\gamma y)^2}$$

The author has not noticed that this formula is not new; it was first given by Ritz, and has been extensively used by him and by Paschen.

In the fourth section, this "new" formula is tested, and found to be fairly satisfactory, though the series chosen (except *Al*) are what one might call "easy" ones; that is, their deviation from the Balmer form is not extreme. Due credit is not given here to the calculations of Birge, who showed that the Ritz formula would do for some of these same series. In the case of *Al*, a "difficult" series is studied, and the formula fails to represent it; but the conclusion is drawn that the first line is improperly classified with the others. More light is needed on this question before a decision can be reached.

Section 5 is largely occupied by a presentation of the graphical methods of study of the series curves. A convenient notation is presented for classifying the series of an element, including those deduced by the "combination-principle" of Ritz. Admirable as such a scheme is, it accounts for far more series than are observed, and fails to account for some (e.g., in *Ca*) that have been found; so that, at first sight, one is tempted to admire it more for its beauty than for its usefulness. Even in the case of the "observed" combination-series, only a few terms have been found, and we are badly in need of more observations to allay the suspicion that these may not be series after all. Mogendorff has recently presented the seductive idea that combination-series are to be regarded as summation- or difference-vibrations (analogous to difference-tones in acoustics) generated by some of the stronger lines of the spectrum. If this can be established as a general principle,

we shall have a much firmer foundation to build a series system upon than is furnished by any approximate formula.

In the last section, the author attempts to group the lines of tin and antimony into series, of which nine are given for the first, twelve for the second element. The reality of some of these series may be questioned, as they are composed of lines of different physical aspects, (e.g., some reversed, others not), and their intensities are not regularly arranged. The problem is a difficult one, as the lines are in the extreme ultra-violet, and more observations are needed, not only of the wave-lengths, but of the Zeeman effect in this region. The author has misunderstood Kayser and Runge's scale of intensities, and has not hesitated to fill a gap in an "arc" series with a "spark" line. It gives rather a shock to one who feels, with Hicks, that "one gets an almost superstitious belief in the measurements of Kayser and Runge after long use of them in this kind of work," to see the author interpret one of their strong reversed lines as a close pair of weak ones. This section of the work would have been improved, and perhaps altered, if the author had made observations directly upon the spectra concerned.

It is a dangerous business to attempt to arrange lines into series by means of a formula alone. A series is an aggregation of lines which are due to the same, or to similar causes, and are affected in similar ways by outside agencies. They must show a progressive arrangement of intensities, and of physical aspect, as well as of frequencies. Since the arrangement of frequencies is exactly given by no known formula (excepting in the case of hydrogen), the physical aspect of the lines becomes all the more important, especially if we include under this head the Zeeman effect, the pressure shift, and the various changes which occur in passing from one sort of source to another, or among different parts of the same source. So far as we know, all the members of a well-behaved series change in the same manner in all these various alterations of conditions, and one cannot help hesitating before accepting a family of new series, until some physical proof of their relationship is presented.

F. A. SAUNDERS

NOTICE

The scope of the ASTROPHYSICAL JOURNAL includes all investigations of radiant energy, whether conducted in the observatory or in the laboratory. The subjects to which special attention is given are photographic and visual observations of the heavenly bodies (other than those pertaining to "astronomy of position"); spectroscopic, photometric, bolometric, and radiometric work of all kinds; descriptions of instruments and apparatus used in such investigations; and theoretical papers bearing on any of these subjects.

Articles written in any language may be accepted for publication, but unless a wish to the contrary is expressed by the author, they usually will be translated into English. Tables of wave-lengths will be printed with the short wave-lengths at the top, and maps of spectra with the red end on the right unless the author requests that the reverse procedure be followed.

Accuracy in the proof is gained by having manuscripts typewritten, provided the author carefully examines the sheets and eliminates any errors introduced by the stenographer. It is suggested that the author should retain a carbon or tissue copy of the manuscript, as it is generally necessary to keep the original manuscript at the editorial office until the article is printed.

All drawings should be carefully made with India ink on stiff paper, usually each on a separate sheet, on about double the scale of the engraving desired. Lettering of diagrams will be done in type around the margins of the cut where feasible. Otherwise printed letters should be put in lightly with pencil, to be later impressed with type at the editorial office, or should be pasted on the drawing where required.

Where an unusual number of illustrations may be required for an article, special arrangements are made whereby the expense is shared by the author or by the institution he represents.

Authors will please carefully follow the style of this JOURNAL in regard to footnotes and references to journals and society publications.

Authors are particularly requested to employ uniformly the metric units of length and mass; the English equivalents may be added if desired.

If a request is sent *with the manuscript*, one hundred reprint copies of each paper, bound in covers, will be furnished free of charge to the author. Additional copies may be obtained at cost price. No reprints can be sent unless a request for them is received before the JOURNAL goes to press.

The editors do not hold themselves responsible for opinions expressed by contributors.

The ASTROPHYSICAL JOURNAL is published during each month except February and August. The annual subscription price is $5.00; postage on foreign subscriptions, 62 cents additional. Business communications should be addressed to *The University of Chicago Press, Chicago, Ill.*

All papers for publication and correspondence relating to contributions should be addressed to *Editors of the* ASTROPHYSICAL JOURNAL, *The University of Chicago, Chicago, Illinois, U.S.A.*

INDEX TO VOLUME XXXVI

SUBJECTS

PAGE

ABSORPTION of Light, Selective, on the Moon's Surface and Lunar Petrography. *R. W. Wood* 75

Spectrum of Tellurium Vapor and the Effect of High Temperature upon It. *E. J. Evans* 228

ACTINOMETRY, Yerkes. *J. A. Parkhurst* 169

ALGOL Variable *RR Draconis*. *Frederick H. Seares* 368

ALKALINE Earth Sulphides, Influence of Temperature upon the Phenomena of Phosphorescence in. *Herbert E. Ives* and *M. Luckiesh* 330

ALUMINIUM, Cadmium, and Zinc, Band Spectra of. *Emily E. Howson* 286

ANGULAR Diameter of Stars by Means of Elliptically Polarized Light, New Means of Determining the. *S. Pokrowsky* 156

ATTRACTION of Sun-Spots for Prominences. *Frederick Slocum* . . 265

BAND Spectra of Aluminium, Cadmium, and Zinc. *Emily E. Howson* 286

CADMIUM, and Zinc, Band Spectra of Aluminium. *Emily E. Howson* 286

CARBON, Deviation from Lambert's Cosine Law of Emission of Tungsten and, at Glowing Temperatures. *A. G. Worthing* . 345

COMET 1911 *c* (Brooks), Photographic Observations of. *E. E. Barnard* 1

CONVERGENCE Frequency in Spectral Series. *Fernando Sanford* . . 255

DARKENING at the Limb in Eclipsing Variables. I, II. *Henry Norris Russell* and *Harlow Shapley* 239, 385

DEVIATION from Lambert's Cosine Law of the Emission from Tungsten and Carbon at Glowing Temperatures. *A. G. Worthing* . . 345

ECLIPSING Variables. I, II. Darkening at the Limb in. *Henry Norris Russell* and *Harlow Shapley* 239, 358

Variables *W Delphini*, *W Ursae Majoris*, and *W Crucis*, Elements of. *Henry Norris Russell* 133

Variables *W Delphini*, *S Cancri*, *SW Cygni*, and *U Cephei*, Elements of. *Harlow Shapley* 269

ELEMENTS of the Eclipsing Variables *W Delphini*, *W Ursae Majoris*, and *W Crucis*. *Henry Norris Russell* 133

of the Eclipsing Variables *W Delphini*, *S Cancri*, *SW Cygni*, and *U Cephei*. *Harlow Shapley* 269

ELLIPTICALLY Polarized Light, New Method of Determining the Angular Diameter of Stars by Means of. *S. Pokrowsky* . . 156

EMISSION from Tungsten and Carbon at Glowing Temperatures, Deviation from Lambert's Cosine Law of. *A. G. Worthing* . 345

PAGE

INFLUENCE of Temperature on the Phenomena of Phosphorescence in the Alkaline Earth Sulphides. *Herbert E. Ives* and *M. Luckiesh* 330

INTEGRATED Spectrum of the Milky Way. *E. A. Fath* 362

LAMBERT'S Cosine Law of the Emission from Tungsten and Carbon at Glowing Temperatures, Deviation from. *A. G. Worthing* . 345

LIGHT, Primary Standard of. *Herbert E. Ives* 322

METALS, Synthetic Development of Radiation Laws for. *Edward P. Hyde* 89

MILKY Way, Integrated Spectrum of the. *E. A. Fath* 362

MOON'S Surface, Selective Absorption of Light on, and Lunar Petrography. *R. W. Wood* 75

NEW Method of Determining the Angular Diameter of Stars by Means of Elliptically Polarized Light. *S. Pokrowsky* 156

Nova Geminorum No. 2, Observations of the Spectrum of. *Walter S. Adams* and *Arnold Kohlschutter* 293

OBSERVATIONS of the Spectrum of *Nova Geminorum* No. 2. *Walter S. Adams* and *Arnold Kohlschutter* 293

ON the Determination of the Orbital Elements of the Eclipsing Variable Stars. II. *Henry Norris Russell* 54

PHOSPHORESCENCE in the Alkaline Earth Sulphides, Influence of Temperature upon the Phenomena of. *Herbert E. Ives* and *M. Luckiesh* 330

PHOTOGRAPHIC Observations of Comet 1911 *c* (Brooks). *E. E. Barnard* 1

PRIMARY Standard of Light. *Herbert E. Ives* 322

PROMINENCES, Attraction of Sun-Spots for. *Frederick Slocum* . . 265

RADIATION Laws for Metals, Synthetic Development of. *Edward P. Hyde* 89

REVIEWS:

P. G. Nutting. *Outlines of Applied Optics* (A. C. Lunn) . . 85

Dr. Moritz von Rohr. *Die optischen Instrumente* (Stanley C. Reese) 263

T. Van Lohuizen. *Bijdraget tot de Kennis van Lijnenspectra* (F. A. Saunders) 409

RR Draconis, Algol Variable. *Frederick H. Seares* 368

SELECTIVE Absorption of Light on the Moon's Surface and Lunar Petrography. *R. W. Wood* 75

SERIES, Spectral, Convergence Frequency in. *Fernando Sanford* . 255

SOLAR Radiation, High-Level Measurement of. *Frank W. Very* . 000

SPECTRA, Band, of Aluminium, Cadmium, and Zinc. *Emily E. Howson* 286

SPECTRAL Series, Convergence Frequency in. *Fernando Sanford* . . 255

SPECTRUM of the Milky Way, Integrated. *E. A. Fath* 362

of *Nova Geminorum* No. 2, Observations of. *Walter S. Adams* and *Arnold Kohlschutter* 293

PAGE

STANDARD of Light, Primary. *Herbert E. Ives* 322

SULPHIDES, Influence of Temperature upon the Phenomena of Phosphorescence in the Alkaline Earth. *Herbert E. Ives* and *M. Luckiesh* 330

SUN-SPOTS, Attraction of, for Prominences. *Frederick Slocum* . . 265

SYNTHETIC Development of Radiation Laws for Metals. *Edward P. Hyde* 89

TELLURIUM Vapor and the Effect of High Temperature upon It, The Absorption Spectrum of. *E. J. Evans* 228

TEMPERATURE, High, The Absorption Spectrum of Tellurium Vapor and the Effect of, upon It. *E. J. Evans* 228

Influence of, upon the Phenomena of Phosphorescence in the Alkaline Earth Sulphides. *Herbert E. Ives* and *M. Luckiesh* . 330

TERTIARY Standards-with the Plane Grating, The Testing and Selection of Standards. *Charles E. St. John* and *L. W. Ware* . . 14

TUNGSTEN and Carbon, Deviation from Lambert's Cosine Law of Emission at Glowing Temperatures. *A. G. Worthing* . . . 345

VARIABLE *RR Draconis*, Algol. *Frederick H. Seares* 368

VARIABLES, Eclipsing, Darkening at the Limb in. I, II. *Henry Norris Russell* and *Harlow Shapley* 239, 385

W Delphini, *S Cancri*, *SW Cygni*, and *U Cephei*, Elements of the. *Harlow Shapley* 269

W Delphini, *W Ursae Majoris*, and *W Crucis*, Elements of Eclipsing. *Henry Norris Russell* 133

YERKES Actinometry. *J. A. Parkhurst* 169

ZINC, Band Spectra of Aluminium, Cadmium, and. *Emily E. Howson* 286

INDEX TO VOLUME XXXVI

AUTHORS

PAGE

ADAMS, WALTER S., and ARNOLD KOHLSCHUTTER. Observations of the Spectrum of *Nova Geminorum* No. 2 293

BARNARD, E. E. Photographic Observations of Comet 1911 *c* (Brooks) 1

EVANS, E. J. The Absorption Spectrum of Tellurium Vapor and the Effect of High Temperature upon It 228

FATH, E. A. The Integrated Spectrum of the Milky Way . . . 362

HOWSON, EMILY E. Band Spectra of Aluminium, Cadmium, and Zinc 286

HYDE, EDWARD P. The Synthetic Development of Radiation Laws for Metals 89

IVES, HERBERT E. The Primary Standard of Light 322

IVES, HERBERT, E., and M. LUCKIESH. Influence of Temperature upon the Phenomena of Phosphorescence in the Alkaline Earth Sulphides 330

KOHLSCHUTTER, ARNOLD, and WALTER S. ADAMS. Observations of the Spectrum of *Nova Geminorum* No. 2 293

LUCKIESH, M., and HERBERT E. IVES. Influence of Temperature upon the Phenomena of Phosphorescence in the Alkaline Earth Sulphides 330

LUNN, A. C. Review of: *Outlines of Applied Optics*, P. G. Nutting . 85

PARKHURST, J. A. Yerkes Actinometry 169

POKROWSKY, S. On a New Method of Determining the Angular Diameters of Stars by Means of Elliptically Polarized Light . 156

REESE, STANLEY C. Review of: *Die optischen Instrumente*, Dr. Moritz von Rohr 263

RUSSELL, HENRY NORRIS. Elements of the Eclipsing Variables, *W Delphini*, *W Ursae Majoris*, and *W Crucis* 133

On the Determination of the Orbital Elements of Eclipsing Variable Stars. II 54

RUSSELL, HENRY NORRIS, and HARLOW SHAPLEY. On Darkening at the Limb in Eclipsing Variables. I, II 239, 385

ST. JOHN, CHARLES E., and L. W. WARE. Tertiary Standards with the Plane Grating, The Testing and Selection of Standards . 14

SAUNDERS, F. A. Review of: *Bijdrage tot de Kennis van Lijnenspectra*, T. Van Lohuizen 409

SANFORD, FERNANDO. On Convergence Frequency in Spectral Series 255

SEARES, FREDERICK H. The Algol Variable *RR Draconis* . . . 368

PAGE

SHAPLEY, HARLOW. Elements of the Eclipsing Variables *W Delphini*, *S Cancri*, *SW Cygni*, and *U Cephei* 269

SHAPLEY, HARLOW, and HENRY NORRIS RUSSELL. On Darkening at the Limb in Eclipsing Variables. I, II 239, 385

SLOCUM, FREDERICK. Attraction of Sun-Spots for Prominences . 265

WORTHING, A. G. On the Deviation from Lambert's Cosine Law of the Emission from Tungsten and Carbon at Glowing Temperatures 345

WARE, L. W., and CHARLES E. ST. JOHN. Tertiary Standards with the Plane Grating, The Testing and Selection of Standards . 14

WOOD, R. W. Selective Absorption of Light on the Moon's Surface and Lunar Petrography 75

Recent Publications

OF

The University of Chicago Press

The Courts, the Constitution, and Parties. Studies in Constitutional History and Politics. By Andrew C. McLaughlin, Professor of History in the University of Chicago.

308 pages, 12mo, cloth; $1.50, postpaid $1.63

A volume of peculiar interest at this time, when the courts and political parties are subject to general criticism. The discussion is especially significant as coming from a lifelong student of constitutional questions, whose work at the University of Michigan, the Carnegie Institution of Washington, and as head of the Department of History in the University of Chicago, is so widely known. The point of view is historical. Though the articles are scientific they are directed to the reader who is interested in public affairs rather than to the professional student. The work consists of five papers, the first of which discusses the power of a court to declare a law unconstitutional. Two of the papers deal with the growth and essential character of political parties, and are followed by one on the history of differing theories of the federal Union. The work concludes with a discussion of the written constitution in some of its historical aspects, taking up the origin of these documents and the problems of their maintenance and interpretation in the development of the modern popular state.

The Mechanistic Conception of Life. Biological Essays by Jacques Loeb, Head of the Department of Experimental Biology, Rockefeller Institute for Medical Research.

238 pages, 12mo, cloth; $1.50, postpaid $1.65

The achievements of Professor Jacques Loeb in the field of experimental biology have made him so widely known as to insure any book of his a large circle of readers. His experimental work at the universities of Chicago and California, as well as in his present position, gives this volume an especial significance.

In his latest work, *The Mechanistic Conception of Life*, Professor Loeb presents many of the current problems in biology,

and discusses the question whether the phenomena of life can be explained by physical and chemical laws. He finds it possible to control by physical or chemical means not merely the processes of reproduction, but also the conduct of animals with reference to environment.

The New York Medical Journal. The profession, as well as everyone interested in biology, will thank the author and the publishers for collecting these essays and placing them before the reading public.

The Chicago Tribune. It is refreshing to contemplate the wonderful exactitude, clearness, honesty, and unanswerable logic of Dr. Loeb.

Heredity and Eugenics. By John M. Coulter, William E. Castle, Edward M. East, William L. Tower, and Charles B. Davenport.

312 pages, 8vo, cloth; $2.50, postpaid $2.70

Leading investigators, representing the University of Chicago, Harvard University, and the Carnegie Institution of Washington, have contributed to this work, in which great care has been taken by each contributor to make clear to the general reader the present position of evolution, experimental results in heredity in connection with both plants and animals, the enormous value of the practical application of these laws in breeding, and human eugenics. The volume is profusely illustrated.

The Nation, New York. "Heredity and Eugenics" may be heartily recommended to readers seeking, as beginners, to get in touch with the discussion of these subjects. . . . . In most of the lec' res there is an admirable reserve, not to say skepticism, in the treatment of large questions which the public is often misled to regard as already and finally settled.

Railway Economics. A Collective Catalogue of Books in Fourteen American Libraries. Prepared by the Bureau of Railway Economics, Washington, D.C.

450 pages, 8vo, cloth; $3.00, postpaid $3.28

Much of the literature relating to railways is widely scattered. The student of railway transportation finds it impossible to secure access to any all-embracing sources of information in any one of even the largest general libraries.

In the present work the Bureau of Railway Economics has undertaken to list the works relating to the economics of railway transportation that are catalogued in thirteen of the principal libraries of this country, together with those in its own collection.

The volume can fairly lay claim to being a representative bibliography in the field of railway economics.

It is confidently believed that the work will prove valuable to all interested in the literature of railway transportation, and particularly to the economist and practical railroad man.

The Evening Post, New York. Of all these invaluable aids to the student which the librarians of the country are producing, this is the most exhaustive and workmanlike of any that has been issued in the economic field.

American Permian Vertebrates. By Samuel Wendell Williston, Professor of Paleontology in the University of Chicago.

152 pages, 39 plates, 8vo, cloth; $2.50, postpaid $2.68

This work comprises a series of monographic studies with briefer notes and descriptions of new or little-known amphibians and reptiles from the Permian deposits of Texas and New Mexico. The material upon which these studies are based was for the most part collected during recent years by field parties from the University of Chicago. The book is offered as a contribution to knowledge on the subject of ancient reptiles and amphibians, with such summaries and definitions—based chiefly on American forms—as our present knowledge permits. The work is illustrated by the author.

Athenaeum. The paleontologist will welcome the work as a solid contribution to our knowledge of a fauna which is of exceptional interest to the student of evolution, inasmuch as it includes forms that help to bridge over some of the differences between reptiles and amphibians.

Morphology of Gymnosperms. By John M. Coulter, Professor of Botany, and Charles J. Chamberlain, Associate Professor of Botany, in the University of Chicago.

470 pages, 462 illustrations, 8vo, cloth; $4.00, postpaid $4.22

This work is a revised and enlarged edition of the book brought out by Professors Coulter and Chamberlain in 1901. Each of the seven great groups is here presented in detail, and a final chapter discusses the problem of phylogeny and points out the evolutionary tendencies. The extinct groups, notably the primitive "seed-ferns," are now included for the first time; and vascular anatomy is fully recognized as a morphological subject of first importance. The entire presentation is thoroughly and system-

atically organized and arranged with a view to the greatest possible clearness. The illustrations are numerous and in large part original.

Nature. The book is an invaluable record, admirably illustrated, of our present knowledge of the older type of seed-plants.

The Historicity of Jesus. A criticism of the contention that Jesus never lived, a statement of the evidence for his existence, an estimate of his relation to Christianity. By Shirley Jackson Case, Assistant Professor of New Testament Interpretation in the University of Chicago.

360 pages, 12mo, cloth; $1.50, postpaid $1.62

Did Jesus ever live, or is he a mythical personage like the deities of Greece and Rome? is to many people a somewhat startling question. But in recent years his actual existence has been vigorously questioned, and the subject is being given wide notice and discussion. The negative opinion has found supporters in America, England, Holland, France, and Germany. To present a complete and unprejudiced statement of the evidence for Jesus' actual existence is the aim of the author of *The Historicity of Jesus.*

The Nation, New York. It is creditable to American scholarship that the first survey of the entire debate should have been made by a representative of an American divinity school, and that the treatment of the question is adequate and fair in the presentation of the arguments on both sides, and marked by discernment both of the underlying principles and the consequences involved for the religious life.

Sociological Study of the Bible, Showing the Development of the Idea of God in Relation to History. By Louis Wallis, formerly Instructor in Sociology in the Ohio State University.

One volume, bound in cloth; $1.50, postpaid $1.68

This book is written on the basis of the modern scientific interpretation of the Bible; but it approaches Bible-study from a new point of view, using the sociological method of research. The ancient Hebrew nation is treated as a social group originating at the point of contact between Amorite city-states and Israelite clans from the Arabian desert. The great struggle within the nation was primarily between the legal usages of the constituent races. This conflict found expression very slowly in terms of

antagonism between the gods of the Israelites and the Amorites. Mr. Wallis' papers on the subject have been appearing for some years in the *American Journal of Sociology;* but they are entirely recast and revised for book publication.

The Dial. A significant and closely reasoned work.

The Scotsman. Whether or not one agrees with Mr. Wallis, the originality and freshness of his views and the marked ability with which they are propounded cannot be denied.

The Minister and the Boy. By Allan Hoben, Associate Professor of Homiletics in the Divinity School of the University of Chicago.

Illustrated, 180 pages, 12mo, cloth; $1.00, postpaid $1.10

From the first chapter on "The Call of Boyhood," through one on "The Approach to Boyhood," the author leads us by suggestion and informing principle to realize the remarkable opportunity afforded by the raw material "boy," on which the minister may work. Later chapters on play and vocational choice show how to train the boy for citizenship, how responsive he is to the right sort of impetus, and how his religious life may indirectly, and even unconsciously, be stimulated by the proper appeal to his manly instincts, while furnishing him with a normal outlet for his natural enthusiasm. The book is practical throughout and each chapter is filled with concrete suggestions which are vitalized by the author's actual experience as a basis.

Mr. Hoben's book is not only enlightening and entertaining, but is wonderfully practical. It should be in the hands of every probation officer, pastor, teacher, and parent.—JUDGE BEN B. LINDSEY.

Scientific Management in the Churches. By Shailer Mathews, Dean of the Divinity School in the University of Chicago.

66 pages, 12mo, cloth; 50 cents, postpaid 55 cents

Why not apply "scientific management" to the varied activities of the church? is the question asked by Dr. Shailer Mathews in his book. The crying need for it is almost startlingly portrayed, and the broad lines of the plan—the adaptation of "scientific management" to an institution such as the church—are inspiring in their suggestiveness.

The book is of the utmost value to all who belong to the active organization in their church—or who are, in the language of the author, "spiritual workmen."

The Independent. A small volume containing matter of large practical value.

The Continent. Dean Mathews' treatment of the subject is practical, and his suggestions are exceedingly useful.

The Ethics of the Old Testament. By Hinckley G. Mitchell, Professor of Hebrew and Old Testament Exegesis in Tufts College.

420 pages, 12mo, cloth; $2.00, postpaid $2.15

The aim of the author is to present a faithful and, as nearly as possible within the limits of a manual, a comprehensive view of the development of ethical ideas among the Hebrew nation. To this end he takes the books, or parts of books, of the Old Testament in the order of their origin, and discusses their teaching, whether direct or indirect, on the duties that men owe to themselves, their families, and the larger world of which they are a part. This is the first volume to be issued in the series of "Handbooks of Ethics and Religion" edited by Shailer Mathews.

Congregationalist and Christian World. It goes far to popularize the results of modern Biblical scholarship, and in general is within the comprehension of ordinary Bible students.

Old Testament Story. Teacher's Manual and Pupil's Notebook. By Charles H. Corbett.

Manual, $1.00, postpaid $1.09; Notebook, 50 cents, postpaid 59 cents

This latest addition to the Constructive Bible Studies, covering the period from Moses to Solomon, is designed for teachers of pupils from ten to twelve years of age corresponding to grades five and six of the public school. The pupil's equipment consists of a loose-leaf notebook containing a page for each lesson. It includes pictures, maps, outlines, paper models, and an occasional written lesson, thereby providing considerable opportunity for handwork.

The Churchman, New York. Teachers will find much that is helpful in Mr. Corbett's manual.

Unity, Chicago. The unique and really valuable part of the Corbett work is his Pupil's Notebook that accompanies it.

American Poems. Selected and edited with illustrative and explanatory notes and a bibliography. By Walter C. Bronson, Litt. D., Professor of English Literature in Brown University.

680 pages, 12mo, cloth; $1.50, postpaid $1.68

The book offers a most carefully chosen and well-balanced presentation of the poetic works of Americans, covering the entire period of our history. For the teacher as well as the student the value of the work is greatly enhanced by the comprehensive Notes, Bibliography, and Indices. It is believed that

the book will have the wide popularity of Professor Bronson's earlier collection, *English Poems*, which has been adopted by all leading American colleges.

The Dial. The resources of the special collections of Brown University have supplied the editor with the best authorities for accurate texts, and have made possible the widest range of selections.

San Francisco Call. A notably successful attempt to produce a useful book, against which the charge of injudicious inclusions and exclusions commonly urged against similar works cannot be made.

English Poems. Selected and edited with illustrative and explanatory notes and bibliographies. By Walter C. Bronson, Litt. D., Professor of English Literature in Brown University.

I. Old English and Middle English Periods
436 pages, 12mo, cloth; School edition, $1.00, postpaid $1.15
Library edition, $1.50, postpaid $1.65

II. The Elizabethan Age and the Puritan Period
550 pages, 12mo, cloth; School edition, $1.00, postpaid $1.15
Library edition, $1.50, postpaid $1.66

III. The Restoration and the Eighteenth Century
541 pages, 12mo, cloth; School edition, $1.00, postpaid $1.15
Library edition, $1.50, postpaid $1.66

IV. The Nineteenth Century
635 pages, 12mo, cloth; School edition, $1.00, postpaid $1.15
Library edition, $1.50, postpaid $1.68

This series of four volumes is intended primarily to afford college classes and general readers a convenient, inexpensive, and scholarly collection of the most important English poetry.

The selections, so far as possible, are complete poems. The notes, though concisely expressed, occupy nearly a hundred pages in each volume. They contain explanations of words and allusions which the average college student might find obscure; statements by the poet or his friends that throw light on the poem; the poet's theory of poetry when this can be given in his own words; quotations which reveal his literary relationships or his methods of work; and extracts from contemporary criticism to show how the poet was received by his own generation. The last-mentioned feature has contributed much to the remarkable success of the series, which is the best general collection of English poetry that has yet been offered at a reasonable price.

The Journal of Education. These volumes are of supreme importance because of their completeness as to material and the scholarly way in which the poems have been annotated.

Questions on Shakespeare. By Albert H. Tolman, Associate Professor of English Literature in the University of Chicago.

The exercises on each play follow a logical order, embracing general questions, questions on individual acts and scenes, character-study, the relation of the play to its sources, and questions concerning the text or meaning.

The questions upon the following comedies are being issued in pamphlet form: A Midsummer-Night's Dream, Much Ado about Nothing, I Henry IV, II Henry IV, The Merchant of Venice, As You Like It, Twelfth Night, The Tempest.

Price 15 cents each, postpaid 17 cents each

Two bound volumes have appeared in this series as follows:

Questions on Shakespeare Part I, Introduction; 75 cents, postpaid 81 cents.

Questions on Shakespeare Part II, contains exercises on the three parts of Henry VI, Richard III, Love's Labour's Lost, The Comedy of Errors, The Two Gentlemen of Verona, A Midsummer-Night's Dream, and all of the Poems with the exception of the Sonnets. $1.00, postpaid $1.09.

Dr. Horace Howard Furness, Editor of the New Variorum Shakespeare. It is fairly astonishing what a deal of information in all departments of Shakespearian study you have compiled, and set forth alluringly—no small consideration where young people are concerned—within so small a space.

The Elementary Course in English. By James Fleming Hosic, Ph.M., Head of the Department of English in the Chicago Teachers College.

152 pages, 16mo, cloth; 75 cents, postpaid 82 cents

The constantly growing use of this book offers convincing testimony to its value as a practical guide for teachers, supervisors, and parents. It presents in outline a working theory of elementary English, with selected references to the recent literature of the subject. In this way the book is especially adapted to individual study and to group discussions in normal schools, teachers' reading circles, teachers' institutes, and parents' associations. The book contains also a suggestive course of study in composition, grammar, word-study, reading, and literature. Definite standards of attainment in these subjects are indicated for each year. Graded lists of material include stories for reading and for telling, poems for study and memorizing, supplementary reading-books classified by subjects, and selected literary studies for higher grades. The Appendix contains a

list of books to be read to the children, a list of verse collections, and a list of prose collections. The author's long connection with the Department of English in the Chicago Teachers College qualifies him to speak with authority on the subject. The book has been endorsed by the *Course of Study*, the official publication of the Chicago public schools.

Educational Review. A good book. . . . . The thoughtful and studious teacher of elementary English will find it full of helpful suggestions and advice.

Agricultural Education in the Public Schools. By Benjamin M. Davis, Professor of Agricultural Education in Miami University.

170 pages, 8vo, cloth; $1.00, postpaid $1.12

In this book Professor Benjamin M. Davis has attacked the problem of the co-ordination of all the agencies now at work on the problem of agricultural education. He has performed a service which will be appreciated by all who have any large knowledge of the problem and of the difficulties which the movement encounters. He has made an effort to canvass the whole field and to give a detailed exposition of the forces employed in building up a rational course of agricultural education. He has presented more fully than anyone else the materials which define the problem and which make it possible for the teacher to meet it intelligently. The annotated bibliography at the end of the book will do much to make the best material available for anyone desiring to get hold of this material through independent study. The book serves, therefore, as a general introduction to the study of agricultural education.

Nature. Professor Davis may be congratulated on a most valuable and thoughtful expert contribution to the literature of his subject.

A Descriptive Catalogue of Manuscripts in the Libraries of the University of Chicago. Prepared by Edgar Johnson Goodspeed, Associate Professor of Biblical and Patristic Greek and Assistant Director of Haskell Oriental Museum.

140 pages, 8vo, cloth; $1.00, postpaid $1.11

This catalogue containing the description of ninety-nine manuscripts in the possession of the University of Chicago was published in connection with the dedication of the Harper

Memorial Library, June 11, 1912. At that time the manuscripts were transferred to permanent quarters in the Manuscript Room of the new Library. Not all the manuscripts of the University are here included, the Greek papyri in Haskell Oriental Museum and the East Indian and other oriental manuscripts being reserved for separate treatment. Most of the manuscripts described in the catalogue came to the University with the Berlin Collection in 1891. The languages of the manuscripts include Latin, Greek, Italian, Spanish, French, German, English, Dutch, Icelandic, Hebrew, and Arabic.

Outlines of Economics, Developed in a Series of Problems. By Members of the Department of Political Economy of the University of Chicago.

160 pages, interleaved, 12mo, cloth; $1.00, postpaid $1.13

This book is an attempt on the part of its authors to make some advances in the direction of improving the current methods of teaching the elementary course in economics. The ideals which have shaped the character of the book are: (1) A belief that the elementary course in economics offers exceptional opportunities for training in thinking and reasoning—a sort of training the importance of which can hardly be exaggerated. The inductive-problem method here used is believed to be the one best adapted to accomplish this end. (2) A desire to connect the theoretical principles of economics with the actual facts and with problems of the business world, and to induce the student to apply his knowledge of that world to the subject of study.

The result is a careful, analytical syllabus of the subjects usually covered in the introductory course, accompanied by some 1,200 questions and problems, designed: (*a*) to afford set problems for written work; (*b*) to guide the student in his reading, while fostering independent thinking; (*c*) to give direction to classroom discussion. It is expected that the *Outlines* will be used in connection with some textbook.

Nation. In their *Outlines of Economics, Developed in a Series of Problems*, three members of the Department of Political Economy in the University of Chicago have performed with remarkable thoroughness and grasp a task of great difficulty. The book consists in the main of sets of searching questions, dealing successively with every phase of the great subject, the order being determined by the attempt of the

authors "not only to link economic theory with descriptive material, but in a measure to build the theory up out of the familiar events of economic life"; an attempt in which, we believe, they have succeeded as completely as the case admits.

Pragmatism and Its Critics. By Addison Webster Moore, Professor of Philosophy in the University of Chicago.

296 pages, 12mo, cloth; $1.25, postpaid $1.36

The general discussion of the philosophical movement known as "pragmatism" has awakened so spirited a controversy that the present volume is especially timely and significant. Professor Moore lays stress on three phases of the question: (1) the historical background of the movement; (2) its relation to the conception of evolution; (3) the social character of pragmatic doctrines. *Pragmatism and Its Critics* is a good example of the modern type of philosophical exposition. It is in a free, conversational style, is profound without being difficult, and simple without sacrificing accuracy.

Philosophical Review. No student is likely to read these chapters without receiving valuable help. The last chapter of this group, "How Ideas Work," is especially noteworthy as containing one of the most attractive and forcible presentations of the pragmatist theory of truth and error.

The Theology of Schleiermacher. By George Cross, Professor of Christian Theology in the Newton Theological Institution.

356 pages, 12mo, cloth; $1.50, postpaid $1.65

Professor Cross's book attempts to introduce the English-speaking student to Schleiermacher himself. It consists principally of a condensed "thought-translation" of his greatest work, *The Christian Faith.* The exposition is introduced by the interesting story of Schleiermacher's life, with emphasis on his religious experience. This is accompanied by a luminous account of the changes in Protestantism that necessitated a reconstruction of its doctrines. The work closes with a critical estimate of Schleiermacher's contribution to the solution of present religious problems, which in the judgment of scholars will stand as an extremely valuable portion of the book. Taken together, the translation, the analysis, and the critical estimate reveal Schleiermacher as a pioneer in modern religious thought.

The Christian World, London. This work is as timely as it is able. It is remarkable that, considering the enormous influence of Schleiermacher on modern theology, the English-speaking world has hitherto had such meager opportunity of studying the man and his teaching.

First-Year Mathematics for Secondary Schools. Second-Year Mathematics for Secondary Schools. By George William Myers, Professor of the Teaching of Mathematics and Astronomy in the College of Education of the University of Chicago, Assisted by the Instructors in Mathematics in the University High School.

First-Year, 378 pages, 12mo, cloth; $1.00, postpaid $1.15
Teacher's Manual; 80 cents, postpaid 89 cents
Second-Year, 296 pages, 12mo, cloth; $1.50, postpaid $1.63

The two texts cover the essentials of what is commonly required of all pupils in the first two years of secondary schools in this country.

Professor Myers has aimed to make the work of the first high-school year connect smoothly and logically with eighth-grade work through both mensuration and general number, rather than with one of these subjects in the first year and the other in the second year. *First-Year Mathematics* is an outgrowth of these two arithmetical topics, developing algebra through quadratics and introducing considerable preliminary work in geometry before the close of the year.

Second-Year Mathematics begins with constructive and inductive geometry and passes rapidly to demonstrative geometry. It is the author's belief that by the employment of algebraic notation and by the continued application of the equation to geometrical matters, the hold on algebra is kept firm until the opportunity arises to develop with profit other algebraic topics. Plane geometry is fully covered.

The first book may be styled algebra with associated arithmetic and geometry; the second, geometry with associated algebra and trigonometry.

The Journal of Education. These books ought to make an epoch in the teaching of mathematics in secondary schools.

Educational Review. These books seem to promise a correlation of elementary algebra, geometry and physics, and may be what the long-suffering teacher of mathematics has hitherto looked for in vain.

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

AGENTS

THE BAKER & TAYLOR COMPANY - - - - - - New York
THE CAMBRIDGE UNIVERSITY PRESS - - - London and Edinburgh
TH. STAUFFER - - - - - - - - - - Leipzig
THE MARUZEN-KABUSHIKI-KAISHA - - - Tokyo, Osaka, and Kyoto

Ohio State 520.5
A8
J8
2004

VOLUME XXXVI NUMBER 5

THE ASTROPHYSICAL JOURNAL

An International Review of Spectroscopy and Astronomical Physics

GENERAL LIBRARY
DEC 30 1912
UNIV. OF MICH.

EDITED BY

GEORGE E. HALE
Mount Wilson Solar Observatory of the Carnegie Institution of Washington

EDWIN B. FROST
Yerkes Observatory of the University of Chicago

HENRY G. GALE
Ryerson Physical Laboratory of the University of Chicago

WITH THE COLLABORATION OF

JOSEPH S. AMES, Johns Hopkins University
ARISTARCH BELOPOLSKY, Observatoire de Poulkova
WILLIAM W. CAMPBELL, Lick Observatory
HENRY CREW, Northwestern University
NILS C. DUNÉR, Astronomiska Observatorium, Upsala
CHARLES FABRY, Université de Marseille
CHARLES S. HASTINGS, Yale University
HEINRICH KAYSER, Universität Bonn
ALBERT A. MICHELSON, University of Chicago
HUGH F. NEWALL, Cambridge University
ERNEST F. NICHOLS, Dartmouth College
ALFRED PEROT, Paris
EDWARD C. PICKERING, Harvard College Observatory
ANNIBALE RICCÒ, Osservatorio di Catania
CARL RUNGE, Universität Göttingen
ARTHUR SCHUSTER, The University, Manchester
KARL SCHWARZSCHILD, Astrophysikalisches Observatorium, Potsdam

---

DECEMBER 1912

CONTENTS

ON THE DEVIATION FROM LAMBERT'S COSINE LAW OF THE EMISSION FROM TUNGSTEN AND CARBON AT GLOWING TEMPERATURES. - - - - - - A. G. WORTHING 345

THE INTEGRATED SPECTRUM OF THE MILKY WAY - - - - - - E. A. FATH 362

THE *ALGOL* VARIABLE *RR DRACONIS* - - - - - - - FREDERICK H. SEARES 368

ON DARKENING AT THE LIMB IN ECLIPSING VARIABLES. II HENRY NORRIS RUSSELL AND HARLOW SHAPLEY 385

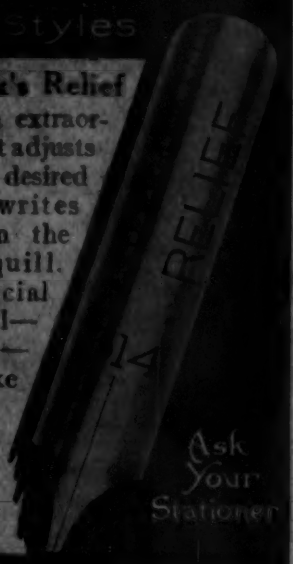
REVIEWS:
Bijdrage tot de Kennis van Lijnenspectra, T. VAN LOHUIZEN (F. A. Saunders), etc.

The University of Chicago Press
CHICAGO, ILLINOIS

AGENTS:
THE CAMBRIDGE UNIVERSITY PRESS, LONDON AND EDINBURGH
WILLIAM WESLEY & SON, LONDON
TH. STAUFFER, LEIPSIG
THE MARUZEN-KABUSHIKI-KAISHA, TOKYO, OSAKA, KYOTO

Esterbrook Steel Pens

250 Styles



Esterbrook's Relief No. 314 is an extraordinary pen that adjusts itself to any desired slant and writes smoother than the old goose quill. Made of special alloyed metal—won't corrode—and finished like a gold pen.

Write for illustrated booklet.

The Esterbrook Steel Pen Mfg. Co.
95 John Street, New York
Works: Camden, N.J.

Ask Your Stationer

FINE INKS AND ADHESIVES
For those who KNOW

Higgins'

- Drawing Inks
- Eternal Writing Ink
- Engrossing Ink
- Taurine Mucilage
- Photo Mounter Paste
- Drawing Board Paste
- Liquid Paste
- Office Paste
- Vegetable Glue, Etc.

Are the Finest and Best Inks and Adhesives

Emancipate yourself from the use of corrosive and ill-smelling inks and adhesives and adopt the Higgins Inks and Adhesives. They will be a revelation to you, they are so sweet, clean, well put up, and withal so efficient.

At Dealers Generally.

CHAS. M. HIGGINS & CO., Mfrs.
Branches: Chicago, London
271 Ninth Street. Brooklyn, N. Y.

ORDERS and inquiries from individual buyers of books in quantities, schools, and libraries solicited. The most prompt and complete shipment of orders, at lowest prices for all parts of the country—the best service in the United States.

Ask us to send you our

STANDARD LIBRARY CATALOGUE NO. 10

and

THE MONTHLY BULLETIN

of the

Latest and Best Selling Books

THE BAKER & TAYLOR COMPANY

Publishers and dealers in the books of all publishers

33-37 East 17th Street NEW YORK CITY Union Square North